FINANCIAL INNOVATION

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Dynamic Hedging Without Probability

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1. Itô scaling and delta hedging can be justified by arbitrage considerations, without probabilistic assumptions. But the justification is asymptotic.

2. With trading frequencies encountered in practice, we need traded instruments to complete the market.

3. Volodya Vovk's methods can be used to compare the effectiveness of different instruments.

I am here to advertise the work of my co-author Volodya Vovk.

Probability and Finance: It's Only a Game!
Wiley 2001.
www.probabilityandfinance.com

Vladimir Vovk atop the World Trade Center in 1999

Vovk's work on game-theoretic probability lays a new foundation for probability and statistics as well as a new approach to option pricing.

Vovk is now completely occupied by advancing applications to statistical prediction.

We need to get him back to work on asset pricing.

Itô scaling

Price changes typically scale like the square root of time.

Kiyosi Itô, at the age of 27, at his desk in the national statistics bureau of Japan.

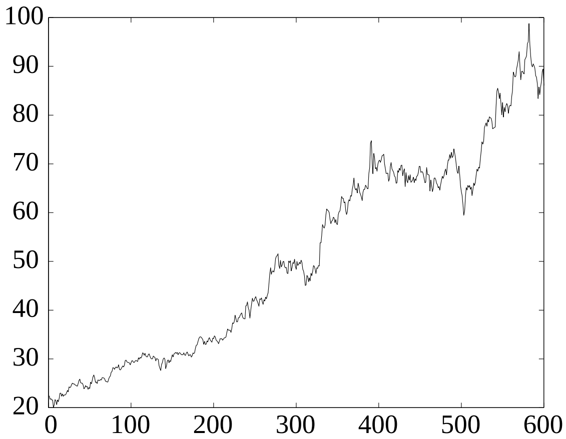
Reference: "A game-theoretic explanation of the $\sqrt{dt}$ effect", by Vovk and Shafer. Working Paper #5 at www.probabilityandfinance.com.

1. What is the $\sqrt{dt}$ effect? **Answer:** When you multiply the time increment δt by K, the typical price change over that increment of time is multiplied by $\sqrt{K}$.

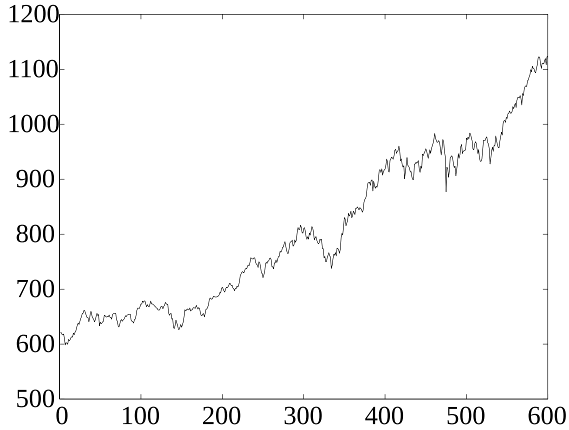
2. Why does the $\sqrt{dt}$ effect happen in markets? **Answer:** Arbitrage. Any different scaling would permit a speculator to make money using a simple momentum or contrarian strategy.

1. What is the $\sqrt{dt}$ effect?

The change dS in a security's price in time dt typically scales like $(dt)^{0.5}$.

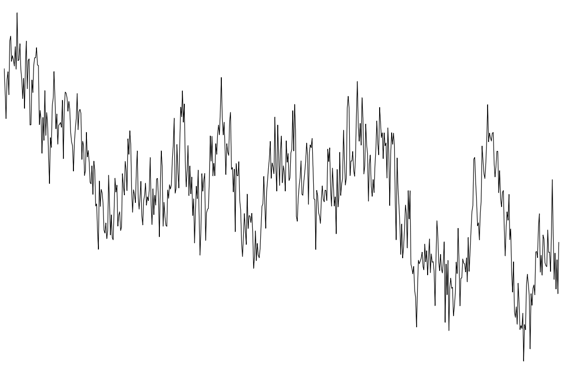


Microsoft, closing prices for 600 working days starting Jan 1, 1996

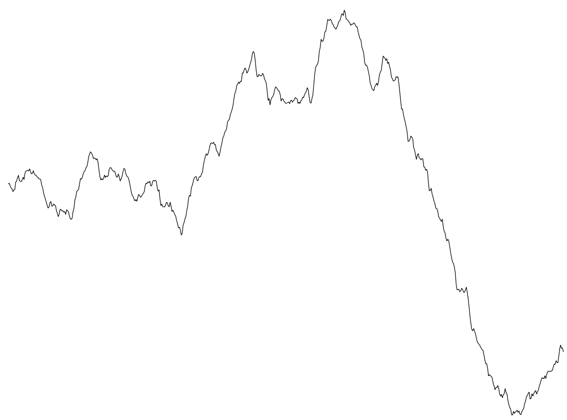


S&P 500, closing values for the same 600 working days

You never get substantially different scaling, as in these artificially generated pictures:



$dS \sim (dt)^{0.2}$



$dS \sim (dt)^{0.8}$

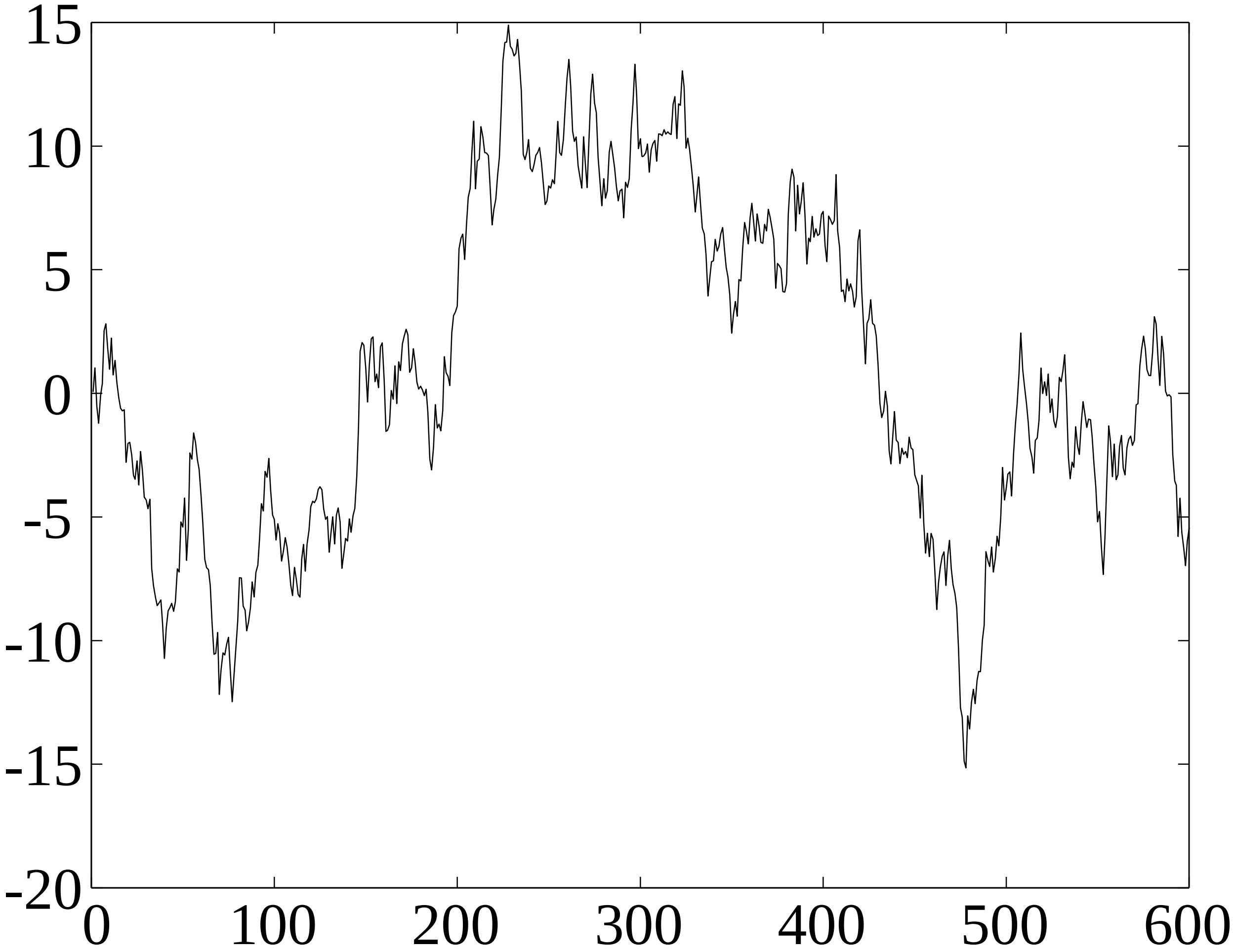
The meaning of the $\sqrt{dt}$ effect lies in the comparison of changes over different time periods.

- If $|dS| \sim (dt)^{0.2}$, daily $|dS|$ is $(1/600)^{0.2} \approx$ 28% of $|dS|$ over 600 days.

- If $|dS| \sim (dt)^{0.5}$, daily $|dS|$ is $(1/600)^{0.5} \approx$ 4% of $|dS|$ over 600 days.

- If $|dS| \sim (dt)^{0.8}$, daily $|dS|$ is $(1/600)^{0.8} \approx$ 0.6% of $|dS|$ over 600 days.

By arbitrary scaling, you can make the daily jump size look the same in all four graphs. But we did not scale arbitrarily. We scaled so that the range of prices over 600 days is about the same for the four graphs.



15
10
5
0
-5
-10
-15
-20
0
100
200
300
400
500
600

The meaning of the $\sqrt{dt}$ effect:

- The average change over one day is about 22% of the average change over one month. ($\sqrt{1/20} \approx 0.22$)

- The average change over one day is about 6% of the average change over one year. ($\sqrt{1/250} \approx 0.06$)

- The average change over one year is about 32% of the average change over ten years. ($\sqrt{1/10} \approx 0.32$)

If the Nikkei 225 has been moving about 1000 points (up or down) each year, you would expect it to move about 60 points each trading day, and about 3000 points over a ten-year period.

When dS scales like $(dt)^H$,

- we call H the *Hölder exponent*,

- we call $p := \frac{1}{H}$ the *variation exponent*, and

- we call $d := 2 - H$ the *box dimension*.

Warning

The meaning of these quantities is extremely asymptotic.

In order to estimate H, p, or d precisely, we need to measure the time series at a very large number of points.

In practice, a financial time series never has more than a few thousand points. So H, p, and d cannot be estimated precisely.

For a continuous price series (an idealization), there are various ways of defining H, p, or d precisely. Today I start by defining p in terms of q-variation. Then I define H by $H := 1/p$.

q-variation

- Divide the time period $[0, T]$ into a large number N of short periods: $dt = \frac{T}{N}$.
- Observe $S_0, S_1, \ldots, S_N$, where S_n is the price at time $ndt = \frac{n}{N}T$.
- Set $dS_n := S_{n+1} - S_n$.
- For $q \geq 0$, the *q-variation* of S is

$$\mathsf{qvar} := \lim_{N \to \infty} \sum_{n=0}^{N-1} |dS_n|^q.$$

Variation exponent

Because $|dS_n| \to 0$ as $N \to \infty$, we expect...

$$\mathsf{qvar} = \infty \quad \text{when} \quad q \approx 0,$$

$$\mathsf{qvar} = 0 \quad \text{when} \quad q >> 1.$$

The *variation exponent* p is the point where qvar changes from being infinite ($q < p$) to being zero ($q > p$).

q-**variation:** For $q \geq 0$, the q-*variation* of S is

$$\mathsf{qvar} := \lim_{N\to\infty} \sum_{n=0}^{N-1} |dS_n|^q.$$

Variation exponent: The variation exponent is the number p such that

$$\mathsf{qvar} = \infty \text{ for } q < p \text{ and } \mathsf{qvar} = 0 \text{ for } q > p.$$

Mathematical difficulty: The limit

$$\lim_{N\to\infty} \sum_{n=0}^{N-1} |dS_n|^q$$

need not exist.

We will resolve this difficulty later in the lecture by substituting an ultraproduct idealization for the usual continuous-function idealization. (This is in the spirit of nonstandard analysis, but simpler.)

Once we assume that the limit always exists, it is clear that qvar must drop abruptly from ∞ to 0 as q increases.

Set

$$\delta_N = \max_{n=0,\ldots,N-1} |dS_n|.$$

By the continuity and boundedness of the price series, the sequence $\delta_1, \delta_2, \ldots$ converges to zero.

So for $r < s$,

$$\sum_{n=0}^{N-1} |dS_n|^s \leq \sum_{n=0}^{N-1} |dS_n|^r \delta_N^{s-r},$$

and since $\lim_{N\to\infty} \delta_N^{s-r} = 0$, we see that $\mathbf{var}_s$ is infinitely smaller than $\mathbf{var}_r$.

For $q \geq 0$, the q-variation is

$$\text{qvar} := \lim_{N \to \infty} \sum_{n=0}^{N-1} |dS_n|^q$$

Intuitively, the *variation exponent* is the point p where qvar drops from ∞ to 0.

We should have $H = 1/p$, because if

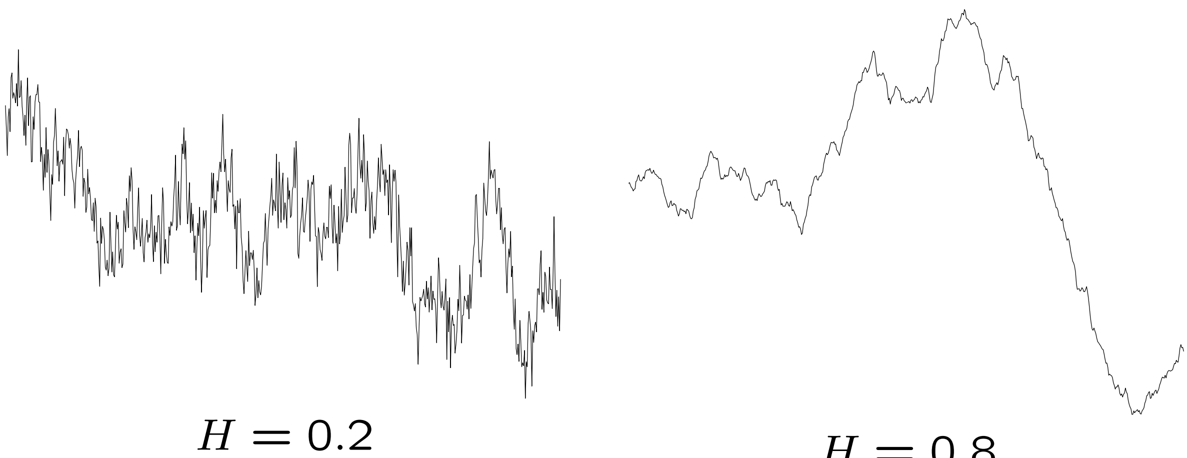
$$dS_n \approx \pm C(dt)^{\frac{1}{p}}$$

for some constant C, then

$$\sum_{n=0}^{N-1} |dS_n|^p \approx N(C(dt)^{\frac{1}{p}})^p \approx C^p N dt = C^p T,$$

which is neither zero nor infinite.

$dS \sim (dt)^{0.2}$ $\qquad$ $dS \sim (dt)^{0.8}$



$H = 0.2$, $p = \frac{1}{H} = 5$ $\qquad$ $H = 0.8$, $p = \frac{1}{H} = 1.25$

The usual measure of variance is

$$\sigma^2 := \frac{1}{T}2\mathsf{var} = \frac{1}{T}\sum_{n=0}^{N-1} |dS_n|^2.$$

If $H = 0.2$, then $dS \approx C(dt)^{0.2}$, and

$$\sigma^2 = \frac{1}{T}\sum_{n=0}^{N-1} |C(dt)^{0.2}|^2$$

$$= \frac{1}{T}N|C(\frac{T}{N})^{0.2}|^2 = T^{-0.6}C^2N^{0.6},$$

which tends to infinity as $N \to \infty$.

If $H = 0.8$, then $dS \approx C(dt)^{0.8}$, and

$$\sigma^2 = \frac{1}{T}\sum_{n=0}^{N-1} |C(dt)^{0.8}|^2$$

$$= \frac{1}{T}N|C(\frac{T}{N})^{0.8}|^2 = T^{0.6}C^2N^{-0.6},$$

which tends to zero as $N \to \infty$.

In theory, the variation exponent p is the value for which

$$\sum_{n=0}^{N-1} |dS_n|^p$$

has a stable value, not tending to zero or infinity as the time period dt is made smaller and smaller.

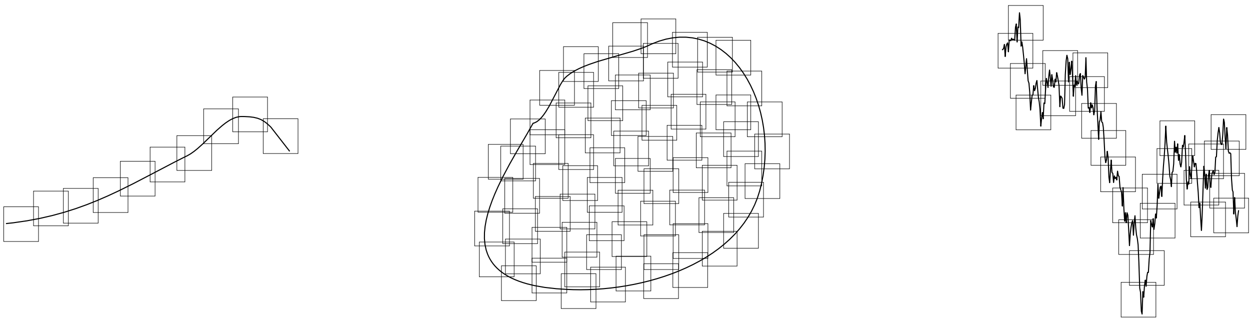
In practice (because we cannot make dt smaller and smaller) the variation exponent is not well defined.

The Itô model, $H = 0.5$ and $p = 2$, usually fits well enough. Mandelbrot has argued, however, that many price series are slightly less jagged—say $H = 0.57$ and $p = 1.75$.

Born in France in 1924, Benoit Mandelbrot spent most of his career at IBM research. He is credited with popularizing fractals in science and among the public. He is an outspoken critic of standard assumptions in finance.

See his *Fractals and Scaling in Finance: Discontinuity, Concentration, Risk*, Springer, 1997.

Mandelbrot's box dimension (just for fun)



The box dimension of an object in the plane is the power to which we should raise $1/dt$ to get the number, up to the order of magnitude, of $dt \times dt$ boxes required to cover it.

Sometimes box dimension is called *fractal dimension*. But is only one of many notions of fractal dimension.

- Object of area A: About $A/(dt)^2$ boxes are required. Box dimension = 2.

- Smooth curve of length T, about T/dt boxes are required. Box dimension = 1.

- Graph of function on $[0, T]$ with Hölder exponent H: We must cover a vertical distance $(dt)^H$ above the typical increment dt on the horizontal axis, which requires $(dt)^H/dt$ boxes. So the number of boxes needed for all T/dt increments is $T(dt)^{H-2}$. Box dimension = $2 - H$.

So the graph of an Itô process has box dimension 1.5.

2. Why does the $\sqrt{dt}$ effect happen?

Answer: Because otherwise a speculator could make a lot of money without risking bankruptcy.

- If prices are more jagged than $\sqrt{dt}$ (daily changes tend to exceed 6% of annual changes), then a simple contrarian strategy can make a lot of money.

- If prices are less jagged than $\sqrt{dt}$ (daily changes tend to be less than 6% of annual changes), then a simple momentum strategy can make a lot of money.

The $\sqrt{dt}$ effect is a consequence of market efficiency. We get $\sqrt{dt}$ when the market blocks speculators who play momentum and contrarian strategies.

Remember what the $\sqrt{dt}$ effect means in practice:

- More jagged than $\sqrt{dt}$ means $\sum_n |dS_n|^2$ is large relative to $\max_n |S_n - S_0|$.

- Less jagged than $\sqrt{dt}$ means $\sum_n |dS_n|^2$ is small relative to $\max_n |S_n - S_0|$.

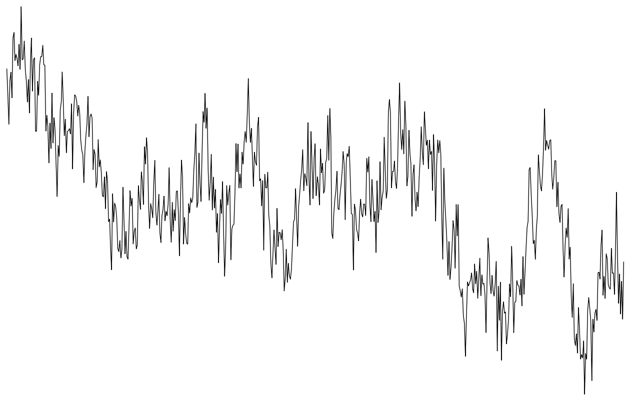
We make our claims precise this way:

- If we can count on $\sum_n |dS_n|^2 \geq \sigma^2_{\text{min}}$ and $\max_n |S_n - S_0| \leq D$, then a simple contrarian strategy can turn \$1 into $\$\sigma^2_{\text{min}}/D^2$ or more for sure.

- If we can count on $\sum_n |dS_n|^2 \leq \sigma^2_{\text{max}}$ and $\max_n |S_n - S_0| \geq D$, then a simple momentum strategy can turn \$1 into $\$D^2/\sigma^2_{\text{max}}$ or more for sure.

Too jagged

$$dS \sim (dt)^{0.2}$$



Recall that if we trade 600 times in a year, then $|dS|$ per trading period will average $(1/600)^{0.2} \approx 28\%$ of $|dS|$ over the year.

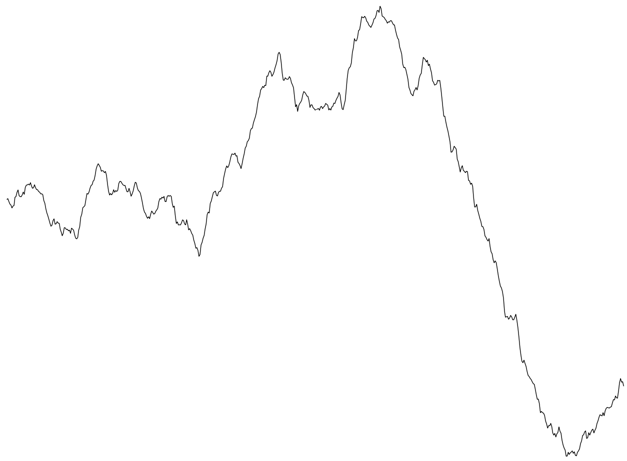
If we can count on $\sum_n |dS_n|^2 \geq \sigma^2_{\text{min}}$ and $\max_n |S_n - S_0| \leq D$, then a simple contrarian strategy can turn \$1 into $\$\sigma^2_{\text{min}}/D^2$ or more for sure.

Say we expect $10 \leq \max_n |S_n - S_0| \leq 20$. Then we might expect trading-period $|dS|$ to average more than 28% of 10, or 2.8, so that $\sum_n |dS_n|^2 \geq 600(2.8)^2 \approx 4700$.

So our contrarian strategy should turn \$1 into about $\$4700/(20)^2 \approx \$12$.

Not jagged enough

$$dS \sim (dt)^{0.8}$$



Recall that if we trade 600 times in a year, then $|dS|$ per trading period will average $(1/600)^{0.8} \approx 0.6\%$ of $|dS|$ over the year.

If we can count on $\sum_n |dS_n|^2 \leq \sigma^2_{\mathsf{max}}$ and $\max_n |S_n - S_0| \geq D$, then a simple momentum strategy can turn \$1 into $\$D^2/\sigma^2_{\mathsf{max}}$ or more for sure.

Say we expect $10 \leq \max_n |S_n - S_0| \leq 20$. Then we might expect trading-period $|dS|$ to average less than 0.6% of 20, or 0.12, so that $\sum_n |dS_n|^2 \leq 600(0.12)^2 \approx 9$.

So our contrarian strategy should turn \$1 into about $\$10^2/9 \approx \$11$.

The claims we want to prove:

- If we can count on $\sum_n |dS_n|^2 \geq \sigma^2_{\text{min}}$ and $\max_n |S_n - S_0| \leq D$, then a simple contrarian strategy can turn \$1 into $\$\sigma^2_{\text{min}}/D^2$ or more for sure.

- If we can count on $\sum_n |dS_n|^2 \leq \sigma^2_{\text{max}}$ and $\max_n |S_n - S_0| \geq D$, then a simple momentum strategy can turn \$1 into $\$D^2/\sigma^2_{\text{max}}$ or more for sure.

We can prove these claims rigorously in a finitary game-theoretic framework, without using any asymptotics or probability.

The Finitary Game-Theoretic Framework

Players: Investor, Market

Protocol:

$S_0 := 0$.

$\mathcal{I}_0 := 1$.

FOR $n = 1, 2, \ldots, N$:

Investor announces $M_n \in \mathbb{R}$.

Market announces $S_n \in \mathbb{R}$.

$\mathcal{I}_n := \mathcal{I}_{n-1} + M_n(S_n - S_{n-1})$.

- $S_0, \ldots, S_N$ is the price process.
- $\mathcal{I}_0, \ldots, \mathcal{I}_N$ is Investor's capital process.
- We assume perfect information; the players move in sequence and everyone sees the moves.

Apology: In most markets, prices are nonnegative. But here, for algebraic simplicity, we follow Bachelier (ordinary Brownian motion) by setting S_0 equal to zero and allowing later S_n to be negative. For the case where the S_n are constrained to be nonnegative, in the spirit of Black-Scholes (or geometric Brownian motion), see our Working Paper #5.

WHEN S_n IS NOT JAGGED ENOUGH...

Players: Investor, Market

Protocol:

$S_0 := 0$.

$\mathcal{I}_0 := 1$.

FOR $n = 1, 2, \ldots, N$:

Investor announces $M_n \in \mathbb{R}$.

Market announces $S_n \in \mathbb{R}$.

$\mathcal{I}_n := \mathcal{I}_{n-1} + M_n(S_n - S_{n-1})$.

Theorem. Suppose Market is constrained to satisfy

- $\sum_n |dS_n|^2 \leq \sigma^2_{\mathrm{max}}$ and
- $\max_n |S_n| \geq D$

for constants $\sigma^2_{\mathrm{max}} > 0$ and $D > 0$ known at the outset of the game. Then Investor has a strategy that guarantees

- $\mathcal{I}_n \geq 0$ for $n = 1, \ldots, N$ and
- $\mathcal{I}_N \geq \frac{D^2}{\sigma^2_{\mathrm{max}}}$.

Lemma. If Investor plays the strategy

$$M_n = 2CS_{n-1},$$

where C is a constant, then

$$\mathcal{I}_n - 1 = C\left(S_n^2 - \sum_{i=0}^{n-1}(dS_i)^2\right). \qquad (1)$$

Proof.

$$\begin{aligned}\mathcal{I}_n - \mathcal{I}_{n-1} &= 2CS_{n-1}\left(S_n - S_{n-1}\right)\\ &= C\left(S_n^2 - S_{n-1}^2 - (S_n - S_{n-1})^2\right).\end{aligned}$$

Proof of theorem. Consider the <u>momentum</u> strategy

$$M_n = 2\frac{1}{\sigma_{\max}^2}S_{n-1}.$$

By Eq. (1),

$$\mathcal{I}_n - 1 = \frac{1}{\sigma_{\max}^2}S_n^2 - \frac{1}{\sigma_{\max}^2}\sum_{i=0}^{n-1}(dS_i)^2 \ge \frac{1}{\sigma_{\max}^2}S_n^2 - 1.$$

Hence

$$\mathcal{I}_n \ge \frac{1}{\sigma_{\max}^2}S_n^2 \ge 0$$

for all n. Let n^* be the first n for which $|S_n| \ge D$. If we stop play at n^* (i.e., play $M_n = 0$ for $n \ge n^*$), we obtain

$$\mathcal{I}_N = \mathcal{I}_{n^*} \ge \frac{D^2}{\sigma_{\max}^2}.$$

WHEN S_n IS TOO JAGGED...

Players: Investor, Market

Protocol:

$S_0 := 0$.

$\mathcal{I}_0 := 1$.

FOR $n = 1, 2, \dots, N$:

Investor announces $M_n \in \mathbb{R}$.

Market announces $S_n \in \mathbb{R}$.

$\mathcal{I}_n := \mathcal{I}_{n-1} + M_n(S_n - S_{n-1})$.

Theorem. Suppose Market is constrained to satisfy

- $\sum_n |dS_n|^2 \geq \sigma^2_{\text{min}}$ and
- $\max_n |S_n| \leq D$

for constants $\sigma^2_{\text{min}} > 0$ and $D > 0$ known at the outset of the game. Then Investor has a strategy that guarantees

- $\mathcal{I}_n \geq 0$ for $n = 1, \dots, N$ and
- $\mathcal{I}_N \geq \frac{\sigma^2_{\text{min}}}{D^2}$.

The Same Lemma. If Investor plays the strategy

$$M_n = 2CS_{n-1},$$

where C is a constant, then

$$\mathcal{I}_n - 1 = C\left(S_n^2 - \sum_{i=0}^{n-1}(dS_i)^2\right).$$

Proof of theorem. Consider the contrarian strategy

$$M_n = -2\frac{1}{D^2}S_{n-1}.$$

By the lemma,

$$\mathcal{I}_n - 1 = \frac{1}{D^2}\sum_{i=0}^{n-1}(dS_i)^2 - \frac{1}{D^2}S_n^2 \geq \frac{1}{D^2}\sum_{i=0}^{n-1}(dS_i)^2 - 1.$$

Hence

$$\mathcal{I}_n \geq \frac{1}{D^2}\sum_{i=0}^{n-1}(dS_i)^2 \geq 0$$

for all n, and

$$\mathcal{I}_N \geq \frac{1}{D^2}\sum_{i=0}^{N-1}(dS_i)^2 \geq \frac{\sigma_{\text{min}}^2}{D^2}.$$