Putting Finance Theory at the Heart of Probability Theory

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- The analogy between probability & finance theory
- Replacing measure theory with game theory.
- The game-theoretic strong law.
- Game-theoretic price and probability.
- The game-theoretic central limit theorem
- The game-theoretic Black-Scholes formula

Probability and Finance: It's Only a Game!
Glenn Shafer and Vladimir Vovk, Wiley 2003.

"The sources of Kolmogorov's *Grundbegriffe*"
Working Paper #4 at www.probabilityandfinance.com
Submitted to *Statistical Science*

PROBABILITY

- **No Dutch book:** Probabilities, interpreted as prices for uncertain payoffs, do not permit you to make money for sure.
- **Cournot's principle:** An event with very small probability, singled out in advance, will not happen.

FINANCE THEORY

- **No arbitrage:** Market prices do not permit you to make money for sure.
- **Efficient markets:** If you do not risk bankruptcy, you definitely will not get rich trading in the market.

We can make the analogy an identity using game theory.

This produces

- a game-theoretic mathematical foundation for probability, more powerful than the measure-theoretic one,
- a clearer philosophy of probability,
- a more realistic theory of finance.

PROBABILITY
Cournot's principle: An event with very small probability, singled out in advance, will not happen.

- The principle, also called the principle of moral certainty, goes back to the ancients, including Cicero.

- Bernoulli (1654–1705) was the first to formulate it within mathematical probability.

- Cournot (1801–1877) was the first to suggest that it provides a bridge between the mathematics of probability and the real world.

- Cournot's idea was refined by Émile Borel (1871–1956), Alexander Chuprov (1874–1926), Paul Lévy (1886-1971), **Andrei Kolmogorov (1903–1987)**, Karl Popper (1902-1994), and many others.

Borel called it the "only law of chance".

Antoine August Cournot (1801–1877). Probabilist, economist, philosopher of science.

Émile Borel (1871–1956). Mathematician, probabilist, politician (Minister of the French Navy in 1925).

Andrei Kolmogorov (1903–1987).Mathematician, survivor. Greatest probabilist of the 20th century.

Volodya Vovk (born 1960). Student of Kolmogorov. Born in Ukraine, educated in Moscow, teaches in London. Volodya is a nickname for the Ukrainian Volodimir and the Russian Vladimir.

PROBABILITY

- **No Dutch book:** Probabilities, interpreted as prices for uncertain payoffs, do not permit you to make money for sure.
- **Cournot's principle:** An event with very small probability, singled out in advance, will not happen.

For the classical probabilists of the early 20th century

- Borel, Fréchet, and Lévy in France,
- Chuprov, Bernstein, and Kolmgorov in Russia,
- Castelnuovo and Cantelli in Italy,

we should distinguish between

- The principle on which you base the mathematics of probability, (either "equally likely cases" or the "no Dutch book principle" or measure theory).
- The principle that gives the mathematical theory meaning in the world, namely Cournot's principle.

Cournot's principle implies that frequency will approximate probability in repeated trials (law of large numbers), that the molecules in the air will not all bunch on one corner of the room (second law of thermodynamics), etc.

FINANCE THEORY
Efficient markets: If you do not risk bankruptcy, you definitely will not get rich trading in the market.

The concept of market efficiency was first discussed by Paul Samuelson and Eugene Fama in the 1960s.

- Samuelson emphasized the connection with the idea of a martingale: "Expected future price must be closely equal to present price, or else present price will be different from what it is. If there were a bargain, which all could recognize, that fact would be 'discounted' in advance and acted upon, thereby raising or lowering present price until the expected discrepancy with future price were sensibly zero..."

- Emphasizing the information traders use, Fama was more explicit about "you cannot get rich".

Everyone now has their own way of explaining the concept of information efficiency in the markets. GREAT CONFUSION.

The confusion is related to the fact that Cournot's principle fell out of the probability literature around 1960, just before Samuelson and Fama were formulating their ideas.

Paul Samuelson (born 1915). Economist who founded modern American finance theory.

Eugene Fama (born 1939). The most frequently cited professor of finance.

- Samuelson: Prices follow a martingale.

- Fama "you can't get rich".

- Principle mathematical fact about a martingale: a non-negative martingale goes to infinity with probability zero.

A martingale is determined by a strategy. The strategy tells how to trade. The martingale gives the resulting wealth.

Both the strategy and the martingale are defined in every situation (defined for every partial sequence of prices).

martingale is non-negative

= strategy does not risk bankruptcy

wealth does not go to infinity

= event of probability zero does not happen

FINANCE THEORY
Efficient markets: If you do not risk bankruptcy, you definitely will not get rich trading in the market.

COURNOT'S PRINCIPLE IN PROBABILITY

- **Finitary form (practical):** An event with very small probability, singled out in advance, will not happen.
- **Infinitary form (theoretical):** An event with zero probability, singled out in advance, will not happen.

EFFICIENT MARKET HYPOTHESIS IN FINANCE THEORY

- **Finitary form (practical):** If you do not risk bankruptcy, you will not multiply your capital by large factor.
- **Infinitary form (theoretical):** If you do not risk bankruptcy, you will not get infinitely rich.

FINANCE THEORY

- **No arbitrage:** Market prices do not permit you to make money for sure.

  This is the basis for pricing options.

- **Efficient markets:** If you do not risk bankruptcy, you definitely will not get rich trading in the market.

  This is the basis for deriving theories that can be tested empirically: securities follow Brownian motion (or at least obey the $\sqrt{dt}$ effect, CAPM, etc.

THE PROJECT: Replace measure theory with game theory as a framework for probability and finance

- Classical theorems in probability become theorems about games where a player may bet on certain specified events at specified odds but *no stochasticity is assumed*.

- No stochastic assumption is needed for option pricing.

- CAPM can be derived with no assumptions of stochasticity and no assumptions about beliefs and preferences of investors.

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Glenn Shafer and Volodya Vovk, Wiley 2001
http://www.cs.rhul.ac.uk/home/vovk/book/

THE PROJECT: Replace measure theory with game theory

MEASURE-THEORETIC FRAMEWORK

- Start with prices for everything.
- Basic framework (measure space) is static. Filtration is added to model time.
- Draw conclusions "except for a set of measure zero" or "with high probability".

GAME-THEORETIC FRAMEWORK

- Limited prices (betting offers).
- Sequential perfect-information game.
- Prices may be given at the outset. Or they may be set in the course of the game!!
- Lower and upper prices can be derived for all payoffs.
- Draw conclusions with high lower probability.

The classical limit theorems (law of large numbers, law of iterated logarithm, central limit theorem) are theorems about a two-player perfect-information game.

On each round of the game:

Player I	(Skeptic)	bets on what Reality will do.
Player II	(Reality)	decides what to do.

Each theorem says that Skeptic has a winning strategy when he is set a certain goal.

Example: Coin Tossing

On each round, Skeptic bets as much as he wants on heads or tails, at even odds. Skeptic wins if (1) he does not go broke, and (2) either he becomes infinitely rich or else the proportion of heads converges to one-half.

Theorem: Skeptic has a winning strategy.

THE STRONG LAW OF LARGE NUMBERS FOR COIN TOSSING

Players: Skeptic, Reality
Protocol:
$\mathcal{K}_0 = 1$.
FOR $n = 1, 2, \ldots$:
Skeptic announces $M_n \in \mathbb{R}$.
Reality announces $x_n \in \{-1, 1\}$.
$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n$.
Winner:
Skeptic wins if
(1) $\mathcal{K}_n$ is never negative and
(2) either $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i = 0$
or else $\lim_{n\to\infty} \mathcal{K}_n = \infty$.
Otherwise Reality wins.

PROPOSITION:

Skeptic has a winning strategy.

Generalize by letting Skeptic choose any number in the interval $[-1, 1]$. Then we get a strong law of large numbers for a bounded sequence of variables $x_1, x_2, \ldots$. (Don't call them "random variables", because they have no probability distribution—just a price of zero on each round.)

> **Players:** Skeptic, Reality
> **Protocol:**
> $\mathcal{K}_0 = 1$.
> FOR $n = 1, 2, \ldots$:
> Skeptic announces $M_n \in \mathbb{R}$.
> Reality announces $x_n \in [-1, 1]$.
> $\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n$.
> **Winner:**
> Skeptic wins if
> (1) $\mathcal{K}_n$ is never negative and
> (2) either $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i = 0$
> or else $\lim_{n\to\infty} \mathcal{K}_n = \infty$.
> Otherwise Reality wins.

PROPOSITION:
Skeptic has a winning strategy.

Generalize further by letting another player (allied with Reality) set the prices on each round.

Players: Skeptic, Reality
Protocol:
$\mathcal{K}_0 = 1$.
FOR $n = 1, 2, \ldots$:
Forecaster announces $m_n \in \mathbb{R}$.
Skeptic announces $M_n \in \mathbb{R}$.
Reality announces $x_n \in [m_n - 1, m_n + 1]$.
$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n(x_n - m_n)$.
Winner:
Skeptic wins if
(1) $\mathcal{K}_n$ is never negative and
(2) either $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}(x_i - m_i) = 0$
or else $\lim_{n\to\infty} \mathcal{K}_n = \infty$.
Otherwise Reality wins.

PROPOSITION:
Skeptic has a winning strategy.

PRICE AND PROBABILITY

$\mathcal{K}_0 := \alpha$.
FOR $n = 1, \ldots, N$:
Skeptic announces $M_n \in \mathbb{R}$.
Reality announces $x_n \in \{-1, 1\}$.
$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n$.

Upper Price for a Variable y:

$\overline{\mathbb{E}}\, y :=$ smallest initial stake Skeptic can parlay into y or more at the end of the game

This is Skeptic's *minimum selling price* for y. He can replicate y at this price with no risk of loss.

Lower Price for a Variable y:

$$\underline{\mathbb{E}}\, y := -\,\overline{\mathbb{E}}[-y].$$

Buying y for α is the same as selling $-y$ for $-\alpha$. So this is Skeptic's *maximum buying price* for y.

$\overline{\mathbb{E}}\, y :=$ smallest initial stake Skeptic can parlay into y or more at the end of the game

Probability from Price

$$\overline{\mathbb{P}}\, E := \overline{\mathbb{E}}\, \mathcal{I}_E \quad \text{and} \quad \underline{\mathbb{P}}\, E := \underline{\mathbb{E}}\, \mathcal{I}_E,$$

where $\mathcal{I}_E$ is the indicator variable for E.

$\overline{\mathbb{P}}\, E =$ smallest stake Skeptic can parlay into at least 1 if E happens and at least 0 otherwise

$=$ inverse of the greatest factor by which Skeptic can multiply his stake if E happens without risking bankruptcy if E fails

If $\overline{\mathbb{P}}\, E = 0$, then Skeptic can get infinitely rich if E happens without risking bankruptcy.

$\mathcal{K}_0 := \alpha$.

FOR $n = 1, \ldots, N$:

Skeptic announces $M_n \in \mathbb{R}$.

Reality announces $x_n \in \{-1, 1\}$.

$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n$.

MARTINGALES

- A *situation* is defined by Reality's moves so far: $x_1, \ldots, x_n$, where $0 \leq n \leq N$. (When $n = 0$, we have the initial situation, which we designate by $\Box$.)

- A *process* is a function on the situations.

- A *martingale* is a capital process for Skeptic (determined by an initial stake α and a strategy for Skeptic).

By the way,

$$\overline{\mathbb{E}}\, y = \inf\{\mathcal{K}(\Box) | \mathcal{K} \text{ is a martingale and } \mathcal{K}(x_1, \ldots, x_N) \geq y(x_1, \ldots, x_N)\}.$$

$\mathcal{K}_0 := \alpha$.
FOR $n = 1, \ldots, N$:
  Skeptic announces $M_n \in \mathbb{R}$.
  Reality announces $x_n \in \{-1, 1\}$.
  $\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n$.

Given a process $\mathcal{P}$, we write $\Delta\mathcal{P}_n$ for its nth increment:

$$\Delta\mathcal{P}_n := \mathcal{P}(x_1, \ldots, x_n) - \mathcal{P}(x_1, \ldots, x_{n-1}).$$

In the case of a martingale $\mathcal{K}$,

$$\Delta\mathcal{K}_n = M_n x_n,$$

where M_n is the move specified by the strategy. Notice that M_n is a function of $x_1, \ldots, x_{n-1}$.

THE CENTRAL LIMIT THEOREM

We consider only coin-tossing (DeMoivre's theorem). For simplicity, we now score Heads as $1/\sqrt{N}$ and Tails as $-1/\sqrt{N}$.

FOR $n = 1, \ldots, N$:
  Skeptic announces $M_n \in \mathbb{R}$.
  Reality announces $x_n \in \{-\frac{1}{\sqrt{N}}, \frac{1}{\sqrt{N}}\}$.
  $\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n$.

Set $\mathcal{S}_n := \sum_{i=1}^{n} x_i$.

Consider a smooth function U.

De Moivre's Theorem For N sufficiently large, both $\overline{\mathbb{E}}\, U(\mathcal{S}_N)$ and $\underline{\mathbb{E}}\, U(\mathcal{S}_N)$ are arbitrarily close to $\int_{-\infty}^{\infty} U(z)\, \mathcal{N}_{0,1}(dz)$.

How do we prove De Moivre's theorem?

$$\mathcal{S}_n := \sum_{i=1}^n x_i.$$

We want to know the price at time 0 of the payoff $U(\mathcal{S}_N)$ at time N. Let us also consider its price at time n. Intuitively, this should depend on $\mathcal{S}_n$, the value of the sum so far. Assume, optimistically, that the price at time n is given by a function of two variables, $\overline{U}(s, D)$: the price at time n is $\overline{U}(\mathcal{S}_n, \frac{N-n}{N})$.

Successive prices are

$$\overline{U}(0,1), \overline{U}(\mathcal{S}_1, \tfrac{N-1}{N}), \ldots$$
$$\ldots, \overline{U}(\mathcal{S}_{N-1}, \tfrac{1}{N}), \overline{U}(\mathcal{S}_N, 0),$$

These must be the successive values of a martingale.

- $\overline{U}(\mathcal{S}_N, 0)$ must equal $U(\mathcal{S}_N)$.
- $\overline{U}(0,1)$ is the price that interests us.

A martingale is a process $\mathcal{K}_n$ with increments of the form $\Delta\mathcal{K}_n = M_n x_n$.

Our task: given U, choose $\overline{U}(s, D)$ so that

(1) $\overline{U}(\mathcal{S}_n, \frac{N-n}{N})$ is a martingale, and

(2) $\overline{U}(\mathcal{S}_N, 0) = U(\mathcal{S}_N)$.

Consider the increments in s, D, and $\overline{U}$:

- $\Delta s_n = x_n = \pm\frac{1}{\sqrt{N}}$.
- $\Delta D_n = -\frac{1}{N}$.
- $\Delta\overline{U}_n = \overline{U}(\mathcal{S}_n, \frac{N-n}{N}) - \overline{U}(\mathcal{S}_{n-1}, \frac{N-n+1}{N})$.

Study $\Delta\overline{U}$ with a Taylor's expansion:

$$\Delta\overline{U} \approx \frac{\partial\overline{U}}{\partial s}\Delta s + \frac{\partial\overline{U}}{\partial D}\Delta D + \frac{1}{2}\frac{\partial^2\overline{U}}{\partial s^2}(\Delta s)^2$$

$$= \frac{\partial\overline{U}}{\partial s}x - \left(\frac{\partial\overline{U}}{\partial D} - \frac{1}{2}\frac{\partial^2\overline{U}}{\partial s^2}\right)\frac{1}{N}.$$

$$\Delta \mathcal{K}_n = M_n x_n$$
$$\Delta \overline{U}_n \approx \frac{\partial \overline{U}}{\partial s} x_n - \left(\frac{\partial \overline{U}}{\partial D} - \frac{1}{2}\frac{\partial^2 \overline{U}}{\partial s^2}\right)\frac{1}{N}$$

We need the second term to go away, which requires

$$\frac{\partial \overline{U}}{\partial D} = \frac{1}{2}\frac{\partial^2 \overline{U}}{\partial s^2}$$

Then we obtain the desired martingale by buying $\frac{\partial \overline{U}}{\partial s}$ x-tickets on the nth round. In other words, we set

$$M_n := \frac{\partial \overline{U}}{\partial s}(\mathcal{S}_{n-1}, \tfrac{N-n+1}{N}).$$

The partial differential equation

$$\frac{\partial \overline{U}}{\partial D} = \frac{1}{2}\frac{\partial^2 \overline{U}}{\partial s^2}$$

is the *heat equation*. Laplace showed that its solution is a Gaussian integral.

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With the initial condition $\overline{U}(s, 0) = U(s)$, the solution is

$$\overline{U}(s, D) = \int_{-\infty}^{\infty} U(z)\, \mathcal{N}_{s,D}(dz)$$

So the initial price, $\overline{U}(0, 1)$, is

$$\int_{-\infty}^{\infty} U(z)\, \mathcal{N}_{0,1}(dz).$$

THE BLACK-SCHOLES PROTOCOL

The price of a security $\mathcal{S}$ is determined by a game just like those we have been studying. If we write $S(t)$ for the price at time t, then we can write the game's protocol as follows.

Parameters: $T > 0$ and $N \in \mathbb{N}$; $dt := T/N$
Players: Investor, Market
Protocol:
  $\mathcal{I}(0) := 0$.
  Market announces $S(0) > 0$.
  FOR $t = 0, dt, 2dt, \ldots, T - dt$:
    Investor announces $\delta(t) \in \mathbb{R}$.
    Market announces $dS(t) \in \mathbb{R}$.
    $S(t + dt) := S(t) + dS(t)$.
    $\mathcal{I}(t + dt) := \mathcal{I}(t) + \delta(t)dS(t)$.

A *European option* on a stock $\mathcal{S}$ with maturity T is a security that pays the amount $U(S(T))$ at time T, where U is a known function. If $\overline{\mathbb{E}}\,U(S(T)) = \underline{\mathbb{E}}\,U(S(T))$, then we say that *the option is priced*.

STOCHASTIC BLACK-SCHOLES

Parameters: $T > 0$ and $N \in \mathbb{N}$; $dt := T/N$
Players: Investor, Market
Protocol:
$\mathcal{I}(0) := 0$.
Market announces $S(0) > 0$.
FOR $t = 0, dt, 2dt, \ldots, T - dt$:
Investor announces $\delta(t) \in \mathbb{R}$.
Market announces $dS(t) \in \mathbb{R}$.
$S(t + dt) := S(t) + dS(t)$.
$\mathcal{I}(t + dt) := \mathcal{I}(t) + \delta(t)dS(t)$.
Constraint on Market: Market must choose $dS(t)$ randomly: $dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$, where $W(t)$ is a standard Brownian motion.

With this constraint on Market, $U(S(T))$ is priced:

$$\overline{\mathbb{E}}\, U(S(T)) = \underline{\mathbb{E}}\, U(S(T)) = \int_{-\infty}^{\infty} U\left(S(0)e^{z}\right) \mathcal{N}_{-\sigma^2 T/2,\, \sigma^2 T}(dz).$$

- **Textbook Stochastic Black-Scholes:**
The market prices the security $\mathcal{S}$. Its price $S(t)$ is assumed to follow geometric Brownian motion; σ^2 can be estimated from past $dS(t)$. All options are priced by plugging the estimate of σ^2 into the Black-Scholes formula.

- **Stochastic Black-Scholes in Practice:**
The market prices $\mathcal{S}$ and also puts/calls on $\mathcal{S}$. (These form a two-dimensional array: a range of strikes and a range of maturities.) Inconsistencies in the put/call prices show that the assumption of geometric Brownian motion for $S(t)$ is faulty (volatility smile). So ad hoc adjustments are required to price other options.

- **Vovk's Game-Theoretic Black-Scholes:**
Instead of pricing puts/calls, the market prices a dividend-paying security $\mathcal{D}$. Each day until maturity, $\mathcal{D}$ pays the dividend $(dS(t)/S(t))^2$. This is only a one-dimensional array: one $\mathcal{D}$ for each maturity. All other options on $\mathcal{S}$ are priced by plugging the market price of $\mathcal{D}$ into the Black-Scholes formula. *No stochastic assumptions or ad hoc adjustments are required.*

Purely Game-Theoretic Black-Scholes

Investor trades in two securities: $\mathcal{S}$, which pays no dividends and $\mathcal{D}$, which pays the dividend $(dS(t)/S(t))^2$.

Parameters: $T > 0$ and $N \in \mathbb{N}$; $dt := T/N$
Players: Investor, Market
Protocol:
Market announces $S(0) > 0$ and $D(0) > 0$.
$\mathcal{I}(0) := 0$.
FOR $t = 0, dt, 2dt, \ldots, T - dt$:
Investor announces $\delta(t) \in \mathbb{R}$ and $\lambda(t) \in \mathbb{R}$.
Market announces $dS(t) \in \mathbb{R}$ and $dD(t) \in \mathbb{R}$.
$S(t + dt) := S(t) + dS(t)$.
$D(t + dt) := D(t) + dD(t)$.
$$\mathcal{I}(t + dt) := \mathcal{I}(t) + \delta(t)dS(t) + \lambda(t)\left(dD(t) + (dS(t)/S(t))^2\right).$$
Constraints on Market: (1) $D(t) > 0$ for $0 < t < T$ and $D(T) = 0$, (2) $S(t) \geq 0$ for all t, and (3) the wildness of Market's moves is constrained.

Once D pays its last dividend, at time T, it is worthless: $D(T) = 0$. So Market is constrained to make his $dD(t)$ add to $-D(0)$.

$$d\mathcal{I}(t) = \delta(t)dS(t) + \lambda(t)\left(dD(t) + (dS(t)/S(t))^2\right)$$

$$d\overline{U}(S(t), D(t)) \approx \frac{\partial \overline{U}}{\partial s}dS(t) + \frac{\partial \overline{U}}{\partial D}dD(t) + \frac{1}{2}\frac{\partial^2 \overline{U}}{\partial s^2}(dS(t))^2$$

Game-theoretic Black-Scholes equation:
We need

$$\delta(t) = \frac{\partial \overline{U}}{\partial s}, \quad \lambda(t) = \frac{\partial \overline{U}}{\partial D}, \quad \frac{\lambda(t)}{S^2(t)} = \frac{1}{2}\frac{\partial^2 \overline{U}}{\partial s^2}.$$

The two equations involving $\lambda(t)$ require that the function $\overline{U}$ satisfy

$$-\frac{\partial \overline{U}}{\partial D} + \frac{1}{2}s^2\frac{\partial^2 \overline{U}}{\partial s^2} = 0$$

for all s and all $D > 0$.

Game-theoretic Black-Scholes formula:
With initial condition $\overline{U}(s, 0) = U(s)$, the solution is

$$\overline{U}(s, D) = \int_{-\infty}^{\infty} U\left(se^z\right) \mathcal{N}_{-D/2,D}(dz).$$

To summarize, the price at time t for the European option $\mathcal{U}$ in a market where both the underlying security $\mathcal{S}$ and a volatility security $\mathcal{D}$ with dividend $(dS(t)/S(t))^2$ are traded is

$$\mathcal{U}(t) = \int_{-\infty}^{\infty} U\left(S(t)e^z\right) \mathcal{N}_{-D(t)/2, D(t)}(dz).$$

To hedge this price, we hold a continuously changing portfolio, containing

$$\frac{\partial \overline{U}}{\partial s}(S(t), D(t)) \text{ shares of } \mathcal{S}$$

and

$$\frac{\partial \overline{U}}{\partial D}(S(t), D(t)) \text{ shares of } \mathcal{D}$$

at time t.

- **Textbook Stochastic Black-Scholes:**
  Assume geometric Brownian motion.
  To price options, plug estimate of σ^2 into Black-Scholes formula.

- **Stochastic Black-Scholes in Practice:**
  Market prices two-dimensional array of options.
  To price other options, make ad hoc adjustments to Black-Scholes formula.

- **Vovk's Game-Theoretic Black-Scholes:**
  Market prices a dividend-paying security $\mathcal{D}$.
  This is only a one-dimensional array.
  To price other options, plug market price of $\mathcal{D}$ into Black-Scholes formula.
  No stochastic assumptions.
  No ad hoc adjuments.

We are calling for a far-reaching change in how option exchanges are organized. The change will be hard to sell and complex to implement but should greatly increase efficiency.