GORENSTEIN LECTURE

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The Empirical Meaning of Probability, from Jakob Bernoulli to the Efficient Market Hypothesis

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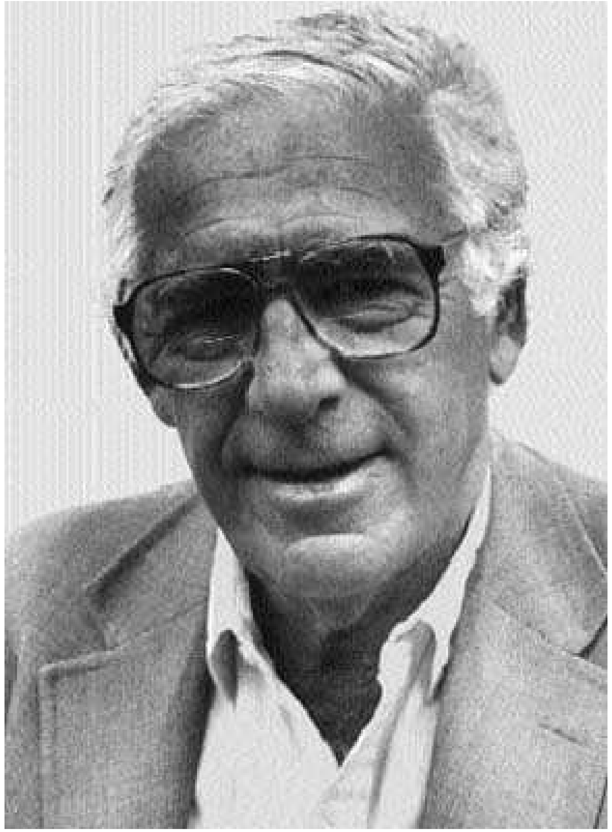


Daniel Gorenstein
1923–1992

The most illustrious of Rutgers mathematicians, Gorenstein finds his place in every index of mathematical personalities.

Today I will speak of some other mathematicians of this caliber.

Jakob Bernoulli (1654–1705)

Antoine Cournot (1801–1877)

Jacques Hadamard (1865-1963)

Émile Borel (1871–1956)

Aleksandr Chuprov (1874–1926)

Andrei Kolmogorov (1903–1987)

Ronald Fisher (1890–1962)

Paul Samuelson (born 1915)

Joseph Doob (born 1920)

Vladimir Vovk (born 1960)

**The Empirical Meaning of Probability,
from Jakob Bernoulli to Paul Samuelson...**

Jakob Bernoulli
1654–1705
An event with very small probability won't happen.

Paul Samuelson
born 1915
You won't make a lot of money without risking bankruptcy.

The Empirical Meaning of Probability, from Jakob Bernoulli to Paul Samuelson...

- In his *Ars Conjectandi* (1713), Bernoulli formulated a fundamental principle connecting probability with the empirical world: **an event with very small probability will not happen**.

- In 1965, Samuelson formulated the efficient market hypothesis: **a speculator will not make a lot of money without risking bankruptcy**.

THESIS OF THIS LECTURE:

Bernoulli's principle is stronger than Samuelson's, but Samuelson's is enough for probability and finance.

Before Bernoulli

Before Bernoulli, probability was not empirical. Earlier authors, such as Pascal, were concerned with equity (option pricing).

- Bernoulli (1713): **An event with very small probability will not happen.**

- Samuelson (1965): **A speculator will not make a lot of money without risking bankruptcy.**

- Bernoulli (1713): **An event with very small probability will not happen.**

Between Bernoulli and Samuelson

Many other accounts of the meaning of probability were given:

Probability = frequency

Probability = belief

Probability theory = measure theory

- Samuelson (1965): **A speculator will not make a lot of money without risking bankruptcy.**

- Bernoulli (1713): **An event with very small probability will not happen.**

- Samuelson (1965): **A speculator will not make a lot of money without risking bankruptcy.**

After Samuelson

Vladimir Vovk's game-theoretic framework for probability and finance clarifies the relationship between Bernoulli's principle and Samuelson's.

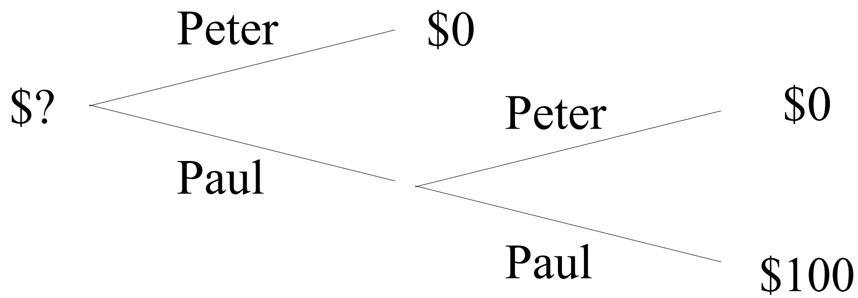
- *Probability and Finance: It's Only a Game!*
  Glenn Shafer and Vladimir Vovk, Wiley, 2001
- www.probabilityandfinance.com

Before Bernoulli, probability was not empirical. . .

Blaise Pascal (1623–1662), as imagined in the 19th century by Hippolyte Flandrin.

Pascal was concerned with the fair division of stakes.

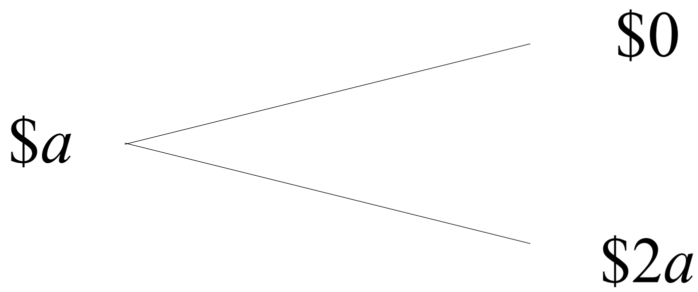
Peter and Paul are playing for $100. Paul is behind. Paul needs 2 points to win, and Peter needs only 1.



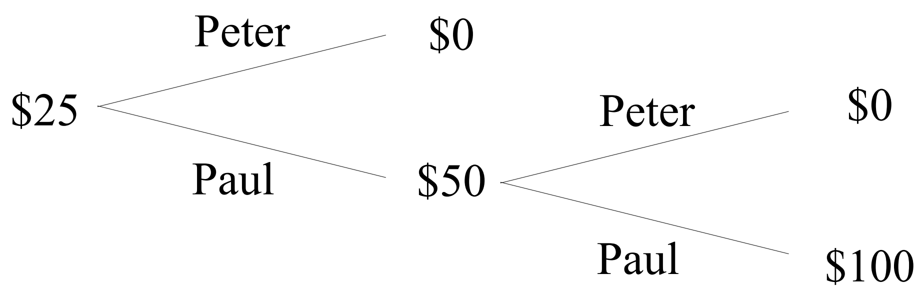
If the game must be broken off, how much of the $100 should Paul get?

It is fair for players to put up equal stakes if the winner takes all.

So it is fair for Paul to pay $\$a$ in order to get $\$2a$ if he defeats Peter and $\$0$ if he loses to Peter.



So Paul should get $25.



Bernoulli introduced mathematical probability is his celebrated Ars Conjectandi, which appeared in 1713.

Jakob Bernoulli
(1654–1705)

"A probability is a degree of certainty and differs from it as a part differs from the whole."

Bernoulli's understanding of the empirical aspect of probability began with the concept of moral certainty.

"Something is *morally certain* if its probability is so close to certainty that the shortfall is imperceptible. Something is *morally impossible* if its probability is no more than the amount by which moral certainty falls short of complete certainty."

How Bernoulli connected probability with the world:

"*Because it is only rarely possible to obtain full certainty, necessity and custom demand that what is merely morally certain be taken as certain.* It would therefore be useful if fixed limits were set for moral certainty by the authority of the magistracy—if it were determined, that is to say, whether 99/100 certainty is sufficient or 999/1000 is required. . ."

In other words, an event with very small probability will not happen.

In the mid-20th century, this became known as "Cournot's principle".

Bernoulli's Theorem:

Fix the level of probability required for moral certainty.

Suppose an event is repeated many times, the odds for its happening always being the same, say p to q.

Let r be the number of times the event happens, s the number of times it fails.

Then it is morally certain that

$$\frac{r}{s} \approx \frac{p}{q}.$$

The larger the number of trials, the better the approximation.

BERNOULLI WAS NOT A FREQUENTIST.

The Frequentist

Richard von Mises
1883–1957

Probability =
limiting frequency

Jakob Bernoulli
1654–1705

It is morally certain that frequencies will approximate probabilities. Many other things are also morally certain.

Who agreed with Bernoulli about moral certainty and moral impossibility?

Most of the great French and Russian probabilists of the 18th, 19th, and early 20th centuries. For example. . .

Antoine Cournot (1801–1877)

Jacques Hadamard (1865-1963)

Émile Borel (1871–1956)

Aleksandr Chuprov (1874–1926)

Andrei Kolmogorov (1903–1987)

Who agreed with Bernoulli about moral certainty and moral impossibility?

Antoine Cournot
1801–1877

Mathematician, economist, philosopher. Rector at Grenoble.

Cournot discussed both *moral impossibility* (very small probability) and *physical impossibility* (infinitely small probability).

> A physically impossible event is one whose probability is infinitely small. This remark alone gives substance—an objective and phenomenological value—to the mathematical theory of probability.

Maurice Fréchet (1878–1973) proposed the name *Cournot's principle* for the principle that an event of small or zero probability will not happen.

Who agreed with Bernoulli about moral certainty and moral impossibility?

Jacques Hadamard
1865-1963

The last universal mathematician.

According to Hadamard, probability theory is based on two principles:

- *The principle of equally likely cases.* This is the basis of the mathematics.
- *The principle of the negligible event.* This connects the mathematics with the real world.

Who agreed with Bernoulli about moral certainty and moral impossibility?

Émile Borel

1871–1956

Inventor of measure theory.

Minister of the French navy in 1925.

Borel was emphatic: the principle that an event with very small probability will not happen is *the only law of chance* (la loi unique du hasard).

- Impossibility on the human scale: $p < 10^{-6}$.
- Impossibility on the terrestrial scale: $p < 10^{-15}$.
- Impossibility on the cosmic scale: $p < 10^{-50}$.

Who agreed with Bernoulli about moral certainty and moral impossibility?

Aleksandr Chuprov
1874–1926

Pre-revolutionary Russian statistician and philosopher of statistics.

Petersburg Polytechnical Institute

Chuprov brought Cournot's principle into Russia and into the theory of statistics.

It was, he said, the basic principle of the logic of probable.

Who agreed with Bernoulli about moral certainty and moral impossibility?

Andrei Kolmogorov
1903–1987

Hailed as the Soviet Euler, Kolmogorov was credited with establishing measure theory as the mathematical foundation for probability.

In his celebrated monograph on measure-theoretic probability (1933), Kolmogorov agreed with his predecessors that Cournot's principle is needed to relate mathematical probability to the real world:

> When P(A) very small, we can be practically certain that the event A will not happen on a single trial of the conditions that define it.

The Measure-Theoretic Framework

In the twentieth century, the classical foundation for probability (equally likely cases + Cournot's principle) was criticized for its lack of mathematical rigor.

Measure theory is rigorous mathematics. A measure P is any set function satisfying

1. P(empty set) = 0, and

2. $\mathsf{P}(A) + \mathsf{P}(B) = \mathsf{P}(A \cup B)$ when A and B are disjoint.

Kolmogorov taught that the rules of measure theory are the proper mathematical foundation for probability.

Although he replaced equally likely cases with measure theory, Kolmogorov stayed close to the classical foundation on the philosophical side.

Kolmogorov's two principles:

A. Principle of Frequency The frequency of an event in a large number of trials will approximate its probability.

B. Cournot's Principle On a single trial, an event of small probability will not happen.

Who disagreed? Who rejected Cournot's principle?

The Germans and the English. For example. . .

- Emmanuel Czuber (1851–1925)
- **Hans Reichenbach (1891–1953)**
- Rudolf Carnap (1891–1970)
- Karl Pearson (1857–1936)
- **Ronald Fisher (1874–1926)**

The Germans had no use for Cournot's principle.

Hans Reichenbach
1891–1953

Reichenbach first wrote on the philosophy of probability in his dissertation in 1915.

Whereas the French and the Russian mathematicians did their own philosophy in the late 18th and early 19th centuries, the Germans had already established a modern division of labor.

For the German philosophers, the guide was Emmanuel Kant, and the probabilistic truths about the world were synthetic. In this optic, Cournot's principle made no sense.

The English had no use for Cournot's principle.

Ronald Fisher
1890–1962

Fisher was the giant of 20th century statistics.

The British statisticians saw little substance in French theorizing. Why start with something purely notional (equally likely cases) and then try to relate it to the world? Start with what you see in the world!

For Fisher, a probability was a relative frequency in a large (infinitely large!) population. End of story.

Why did Cournot's priniciple disappear in the second half of the 20th century?

Two monsters:

- Adolf Hitler (1889–1945)

- Joseph Stalin (1879–1953)

A great mathematician:

- Joseph Doob (born 1910)

The great destroyer

Adolf Hitler
1889–1945

World War II and the Holocaust destroyed the primacy of the Parisian school of probability.

Like most of the philosophers Hitler drove out of Vienna in the 1930s, Reichenbach and Carnap settled in the USA. There they set the framework for postwar philosophy of probability.

The great silencer

Joseph Stalin
1879–1953

How to connect probability theory with the real world (statistics) was a dangerous topic under Stalin.

Kolmogorov always expressed his philosophical views tersely, and his western readers often conclude that he had no philosophy.

Probability is measure theory, and there is nothing more to say.

The champion of measure theory

Joseph Doob, born 1910, receiving the National Medal of Science from President Carter in 1979.

Picking up where Kolmogorov left off, Doob showed how probabilities for continuous random processes (e.g., Brownian motion) can be put in the measure-theoretic framework.

Martingales

One of Doob's great contributions was to fit martingales into the measure-theoretic framework.

Intuitively, a martingale is the path followed by a gambler's wealth as he makes successive bets.

Doob showed that if the gambler does not risk bankruptcy, then the martingale becomes infinitely large with probability zero.

Intuitively,

$\boxed{\text{event of probability zero will not happen}} \Leftrightarrow \boxed{\text{martingale will not become infinitely large}}$

Doob's problem

The traditional philosophical foundation for probability (Bernoulli's theorem + Cournot's principle) breaks down for stochastic processes.

Bernoulli's theorem does not apply because we are not repeating the same random experiment over and over.

Doob resolved the problem by belittling philosophy altogether.

Looking ahead to the game-theoretic solution. . .

Vladimir Vovk, born 1960, atop the World Trade Center in 1998.

In the 1990s, Vovk solved Doob's problem by replacing measure theory with game theory.

The principle that the gambler will not get very (or infinitely) rich without risking bankruptcy replaces the old principle that an event of small (or zero) probability will not happen.

Probability and Finance: It's Only a Game! Shafer and Vovk, 2001.

In 1965, Samuelson applied Doob's theory of martingales to market prices.

Paul Samuelson
(born 1915)

Samuelson's efficient market hypothesis: Present price is the expected value of future price.

It follows that the price of an asset is a martingale.

The market takes all information into account in calculating the expected value. So if there were good reason to think the price will be different in the future, it would already be different.

Different meanings of "market efficiency"

1. You will not get rich without risking bankruptcy.

2. Any change in an asset's price implies new information about the asset.

3. The market directs capital to its optimal use.

These different meanings are held together by Doob's measure-theoretic picture.

The different meanings are held together by Doob's measure-theoretic picture.

1. You will not get rich without risking bankruptcy. *Because an event of zero probability will not happen.*

2. Any change in an asset's price implies new information about the asset. *Because the current price is always the expected value based on all current information.*

3. The market directs capital to its optimal use based on current information. *An opportunity to use capital better is an opportunity to make money relative to the market, and this does not exist if price is the expected value of future value given all current information.*

The debate on the efficient market hypothesis continues (Journal of Economic Perspectives, 2003)

For the hypothesis. . .

Burton Malkiel (born 1932)

Malkiel believes the hypothesis has been vindicated by evidence that people cannot make money without undue risk.

Against the hypothesis. . .

Robert Shiller (born 1946)

Shiller believes the hypothesis has been refuted by evidence that many changes in market prices are not based on new information.

In Vovk's game-theoretic framework for probability and finance, Malkiel and Shiller can both be right.

1. You will not get rich without risking bankruptcy. **As Malkiel points out, the evidence for this remains good.**

2. Every change in an asset's price results from new information about the asset. **As Shiller points out, there is good evidence against this.**

In the game-theoretic framework, we do not necessarily have a global probability measure. We have only the market prices. We can adopt hypothesis 1 directly without adopting hypothesis 2.

Conclusions

1. When you assume a global probability measure, Vovk's game-theoretic efficient market hypothesis is equivalent to Cournot's classical principle.

2. Under more realistic assumptions (as when you have only changing market prices, for example), Vovk's hypothesis takes on an independent life.

3. Though more realistic, Vovk's approach preserves classical results in probability theory (e.g., the limit theorems) and finance theory (e.g., the $\sqrt{dt}$ effect and CAPM).