How to invent game-theoretic statistics yourself, in 16 mostly easy steps

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These instructions could be improved, but it would be best not to improve them too much.

Elementary probability

1. Fix a natural number N and a finite set $\mathcal{Y}$. Suppose $Y_1, \ldots, Y_N$ are unknown elements of $\mathcal{Y}$; call them *local variables*. A *probability forecasting system* is a distribution for Y_1 and a distribution for each Y_n for each possible sequence of preceding outcomes $y_1, \ldots, y_{n-1}$. Show how to construct a joint distribution for $Y_1, \ldots, Y_N$ from a probability forecasting system.[1]

2. Put the joint distribution aside and use the probability forecasting system in a different way. Namely, define the concept of a *martingale*: a capital process determined by some initial capital and a strategy for betting on successive steps using as prices the expected values given by the probability forecasting system.

3. Using the concept of a martingale, define the *expected value* of a *global variable* X (i.e., a function of $y_1, \ldots, y_N$) as the initial capital for which there exists a martingale that begins with this initial capital and ends with X.[2] Verify that the initial capital, say $\mathbb{E}(X)$, and the martingale are unique. Call the martingale the *expectation martingale for* X.

4. Verify that the expected values defined in Step 3 agree with those given by the joint distribution constructed in Step 1.

5. Introduce the concept of *probability* in the usual way: the probability $\mathbb{P}(E)$ of a *global event* E (i.e., a subset of the set of possible values for the sequence $y_1, \ldots, y_N$) is the expected value of the indicator variable $\mathbf{1}_E$.

Elementary mathematical statistics

6. Call a nonnegative martingale starting at 1 a *test martingale*. When you are using a test martingale $\mathcal{M}$ to test the probability forecasting system, and you have observed $y_1, \ldots, y_n$ so far, call $\mathcal{M}_n(y_1, \ldots, y_n)$ a *betting score* and interpret it as a measure of the evidence so far against the probability forecasting system.

7. Consider a global event E with probability α. When you decide in advance to interpret the happening of E as evidence against the probability forecasting system, say that you are making *a level-α test with rejection region E*.

[1] Now you have invented classical axiomatic probability theory, in which the main axioms were the rule of total probability and the rule of compound probability.

[2] Now you have invented Pascal's and Huygens's theory of expectation.

8. Make level-α testing (Step 7) a special case of testing with a test martingale (Step 6) by saying that *the level-α test E is implemented by the test martingale $\mathcal{M}$*, where $\mathcal{M}$ is the expectation martingale for the global variable X given by

$$X(y_1, \ldots, y_N) := \begin{cases} \frac{1}{\alpha} & \text{if } (y_1, \ldots, y_N) \in E, \\ 0 & \text{if } (y_1, \ldots, y_N) \notin E. \end{cases}$$

Verify that $\mathbb{E}(\mathcal{M}) = 1$, so that $\mathcal{M}$ is indeed a test martingale. Verify that $\mathcal{M}_n(y_1, \ldots, y_n) \leq 1/\alpha$ for all n and all $y_1, \ldots, y_n$.

9. Consider a level-α test E implemented by a test martingale $\mathcal{M}$. For each n, set

$$E_n = \{(y_1, \ldots, y_n) | \mathcal{M}_n(y_1, \ldots, y_n) = 1/\alpha\}.$$

Call the sequence $E_1, \ldots, E_N$ the *sequential form of the test E*. The statistician can stop observing the sequence and reject the probability forecasting system as soon as one of the E_n happens.[3]

10. Suppose $(\mathcal{FS}^\theta)_{\theta \in \Theta}$ is an indexed class of probability forecasting systems, all with the same N and $\mathcal{Y}$. Suppose $\mathcal{M}^\theta$, for each θ, is a test martingale for $\mathcal{FS}^\theta$.

    (a) Call $\{\theta | \mathcal{M}_N^\theta \geq 1/\alpha\}$, which depends on $y_1, \ldots, y_N$, a $1/\alpha$-*discredit set*, because these observations have discredited each of the corresponding probability forecasting systems at level $1/\alpha$ or more.

    (b) If you believe that $(\mathcal{FS}^\theta)_{\theta \in \Theta}$ includes a probability forecasting system that is reliable,[4] then call the complement of the discredit set, $\{\theta | \mathcal{M}_N^\theta < 1/\alpha\}$, a $1/\alpha$-*warranty set*.[5]

    (c) Show that for a given class $(\mathcal{FS}^\theta, \mathcal{M}^\theta)_{\theta \in \Theta}$, the $1/\alpha$-warranty sets for different α are nested.[6]

11. Consider again an indexed class of probability forecasting systems with associated test martingales, $(\mathcal{FS}^\theta, \mathcal{M}^\theta)_{\theta \in \Theta}$. Set

$$W_n := \{\theta | \mathcal{M}_N^\theta < 1/\alpha\}$$

and note that W_n depends on $y_1, \ldots, y_n$. Verify that $\mathbb{P}^\theta(\theta \in W_n \text{ for all } n) \geq 1 - \alpha$ for all θ.[7]

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[3] Now you have invented a simple case of Wald's sequential analysis.

[4] The notion of reliability used here is discussed on p. 197 of *Game-Theoretic Foundations for Probability and Finance*.

[5] Now you have invented confidence intervals.

[6] Suppose $\alpha_1 < \alpha_2$, and suppose that for all θ, E_1^θ and E_2^θ are global events such that $\mathbb{P}^\theta(E_1^\theta) = \alpha_1$ and $\mathbb{P}^\theta(E_2^\theta) = \alpha_2$. For all θ and $i = 1, 2$, write $\mathcal{M}^{\theta,i}$ for the expectation martingale that implements the level-α_i test E_i^θ. Then the $1/\alpha_1$-warranty set for $(\mathcal{FS}^\theta, \mathcal{M}^{\theta,1})_{\theta \in \Theta}$ contains the $1/\alpha_2$-warranty set for $(\mathcal{FS}^\theta, \mathcal{M}^{\theta,2})_{\theta \in \Theta}$. Should this nesting be interpreted in the same way as the nesting of the $1/\alpha$-warranty sets for $(\mathcal{FS}^\theta, \mathcal{M}^\theta)_{\theta \in \Theta}$?

[7] Now you have invented confidence sequences.

Testing probability forecasts

12. Suppose now that instead of being given a probability forecasting system, you participate in a probability forecasting game. On each of N successive rounds, you are given a probability distribution P on $\mathcal{Y}$ and allowed to bet on the outcome of the round by using the expected values given by P. Generalize elementary probability and statistics to this picture by generalizing the definition of martingale in Step 2; now a martingale is a capital process determined by (1) an initial capital and (2) a strategy for betting on successive rounds using the expected values given by the announced Ps as prices.

Testing incomplete probability forecasts

13. Suppose now that instead of being given a probability distribution P on $\mathcal{Y}$ on each round and allowed to bet using any expected values given by P, you are given fewer betting offers. Generalize to this picture by generalizing the definition of martingale in Step 12; now a *supermartingale* is a capital process determined by (1) an initial capital and (2) a strategy for betting on successive rounds by selecting from the bets offered.[8]

Discrete-time advanced probability

14. Now generalize game-theoretic probability and statistics to the case where the individual distributions in the probability forecasting are not necessarily discrete and the game many continue for an infinite number of rounds.

Point processes

15. Now generalize game-theoretic probability and statistics to cover the case of point processes in continuous time, such as the processes studied in survival analysis.[9]

Continuous processes

16. Now give a game-theoretic account of Brownian motion and other continuous-time stochastic processes.[10]

Conclusion Now that you have invented game-theoretic probability and statistics, you may want to think about what you can do with them.

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[8] Now you have invented game-theoretic imprecise probability.

[9] Congratulations! Now you have invented something no one else has invented before.

[10] You may want to compare your formulation to Vladimir Vovk's formulation, as reported in Chapters 13–17 of *Game-Theoretic Foundations for Probability and Finance.*