Statistics Seminar, University of Tokyo

Tuesday, June 1, 2004

Why do price series look like Itô processes?

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This document has been designed to be viewed two pages at a time—pp. 2-3 together, pp. 4-5 together, and so on. (Select "Continuous - Facing" from the "View" menu in Adobe Acrobat Reader.)

Why do price series look like Itô processes?

Price changes typically scale like the square root of time. This is due to arbitrage. Any different scaling would permit a speculator to make money using a simple momentum or contrarian strategy.

Kiyosi Itô, at the age of 27, at his desk in the national statistics bureau of Japan.

Reference: "A game-theoretic explanation of the $\sqrt{dt}$ effect", by Vladimir Vovk and Glenn Shafer. Working Paper #5 at www.probabilityandfinance.com.

Outline of Lecture

1. What is the $\sqrt{dt}$ effect?

2. Why does the $\sqrt{dt}$ effect happen in markets?

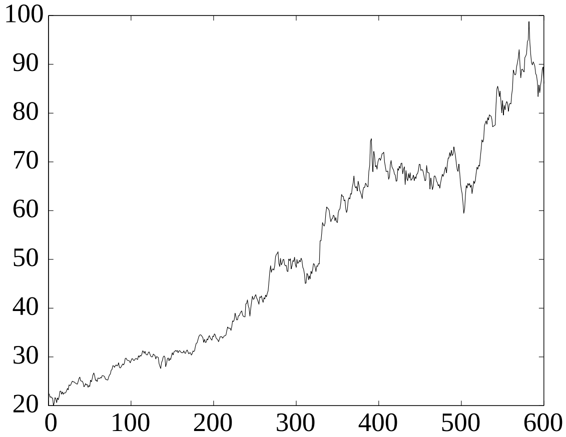
3. Making the picture infinitary using an ultraproduct of games

4. Making the picture infinitary using measure-theoretic probability

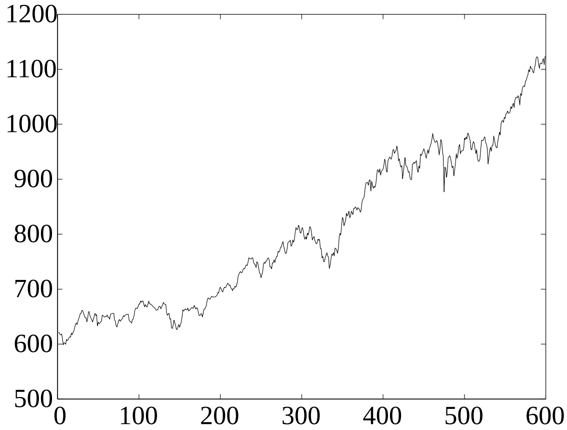
5. Advantages of the ultraproduct picture over the measure-theoretic picture

1. What is the $\sqrt{dt}$ effect?

The change dS in a security's price in time dt typically scales like $(dt)^{0.5}$.

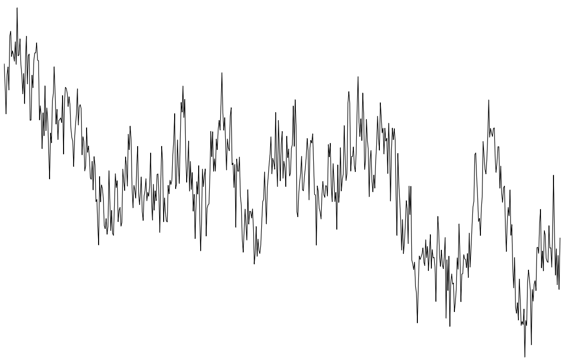


Microsoft, closing prices for 600 working days starting Jan 1, 1996

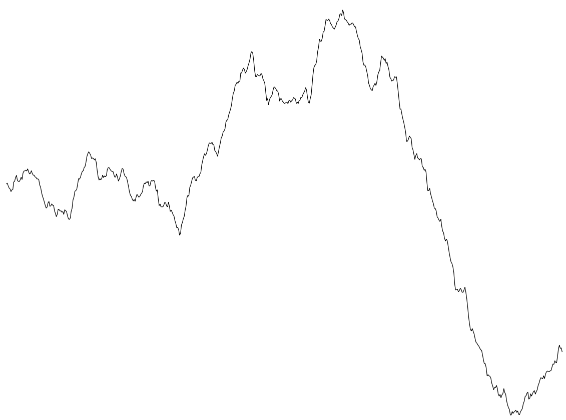


S&P 500, closing values for the same 600 working days

You never get substantially different scaling, as in these artificially generated pictures:



$dS \sim (dt)^{0.2}$



$dS \sim (dt)^{0.8}$

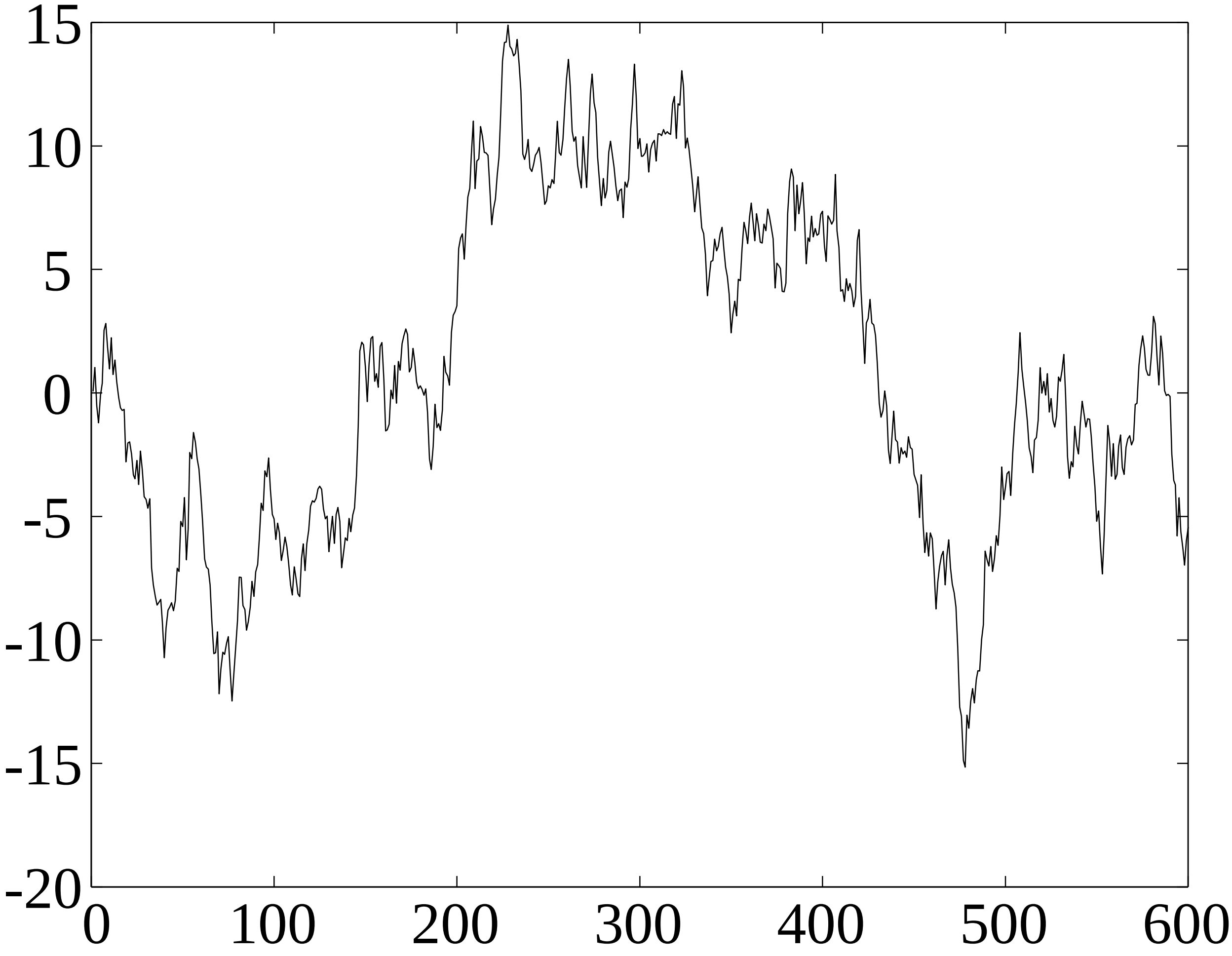
The meaning of the $\sqrt{dt}$ effect lies in the comparison of changes over different time periods.

- If $|dS| \sim (dt)^{0.2}$, daily $|dS|$ is $(1/600)^{0.2} \approx$ 28% of $|dS|$ over 600 days.

- If $|dS| \sim (dt)^{0.5}$, daily $|dS|$ is $(1/600)^{0.5} \approx$ 4% of $|dS|$ over 600 days.

- If $|dS| \sim (dt)^{0.8}$, daily $|dS|$ is $(1/600)^{0.8} \approx$ 0.6% of $|dS|$ over 600 days.

By arbitrary scaling, you can make the daily jump size look the same in all four graphs. But we did not scale arbitrarily. We scaled so that the range of prices over 600 days is about the same for the four graphs.


15
10
5
0
-5
-10
-15
-20
0
100
200
300
400
500
600

The meaning of the $\sqrt{dt}$ effect:

- The average change over one day is about 22% of the average change over one month. ($\sqrt{1/20} \approx 0.22$)

- The average change over one day is about 6% of the average change over one year. ($\sqrt{1/250} \approx 0.06$)

- The average change over one year is about 32% of the average change over ten years. ($\sqrt{1/10} \approx 0.32$)

If the Nikkei 225 has been moving about 1000 points (up or down) each year, you would expect it to move about 60 points each trading day, and about 3000 points over a ten-year period.

When dS scales like $(dt)^H$,

- we call H the *Hölder exponent*,
- we call $p := \frac{1}{H}$ the *variation exponent*, and
- we call $d := 2 - H$ the *box dimension*.

Warning

The meaning of these quantities is extremely asymptotic.

In order to estimate H, p, or d precisely, we need to measure the time series at a very large number of points.

In practice, a financial time series never has more than a few thousand points. So H, p, and d cannot be estimated precisely.

For a continuous price series (an idealization), there are various ways of defining H, p, or d precisely. Today I start by defining p in terms of q-variation. Then I define H by $H := 1/p$.

q-variation

- Divide the time period $[0, T]$ into a large number N of short periods: $dt = \frac{T}{N}$.
- Observe $S_0, S_1, \ldots, S_N$, where S_n is the price at time $ndt = \frac{n}{N}T$.
- Set $dS_n := S_{n+1} - S_n$.
- For $q \geq 0$, the *q-variation* of S is

$$\mathsf{qvar} := \lim_{N \to \infty} \sum_{n=0}^{N-1} |dS_n|^q.$$

Variation exponent

Because $|dS_n| \to 0$ as $N \to \infty$, we expect...

$$\mathsf{qvar} = \infty \quad \text{when} \quad q \approx 0,$$

$$\mathsf{qvar} = 0 \quad \text{when} \quad q >> 1.$$

The *variation exponent* p is the point where qvar changes from being infinite ($q < p$) to being zero ($q > p$).

q-variation: For $q \geq 0$, the *q-variation* of S is

$$\mathsf{qvar} := \lim_{N\to\infty} \sum_{n=0}^{N-1} |dS_n|^q.$$

Variation exponent: The variation exponent is the number p such that

$$\mathsf{qvar} = \infty \text{ for } q < p \text{ and } \mathsf{qvar} = 0 \text{ for } q > p.$$

Mathematical difficulty: The limit

$$\lim_{N\to\infty} \sum_{n=0}^{N-1} |dS_n|^q$$

need not exist.

We will resolve this difficulty later in the lecture by substituting an ultraproduct idealization for the usual continuous-function idealization. (This is in the spirit of nonstandard analysis, but simpler.)

Once we assume that the limit always exists, it is clear that qvar must drop abruptly from ∞ to 0 as q increases.

Set

$$\delta_N = \max_{n=0,\ldots,N-1} |dS_n|.$$

By the continuity and boundedness of the price series, the sequence $\delta_1, \delta_2, \ldots$ converges to zero.

So for $r < s$,

$$\sum_{n=0}^{N-1} |dS_n|^s \leq \sum_{n=0}^{N-1} |dS_n|^r \delta_N^{s-r},$$

and since $\lim_{N\to\infty} \delta_N^{s-r} = 0$, we see that $\mathbf{var}_s$ is infinitely smaller than $\mathbf{var}_r$.

For $q \geq 0$, the q-variation is

$$\text{qvar} := \lim_{N \to \infty} \sum_{n=0}^{N-1} |dS_n|^q$$

Intuitively, the *variation exponent* is the point p where qvar drops from ∞ to 0.

We should have $H = 1/p$, because if

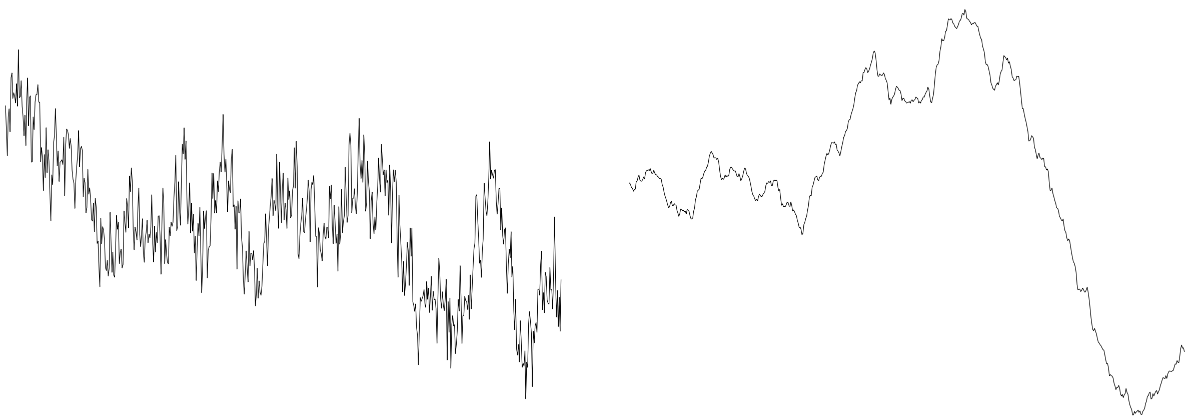
$$dS_n \approx \pm C(dt)^{\frac{1}{p}}$$

for some constant C, then

$$\sum_{n=0}^{N-1} |dS_n|^p \approx N(C(dt)^{\frac{1}{p}})^p \approx C^p N dt = C^p T,$$

which is neither zero nor infinite.

$dS \sim (dt)^{0.2}$ $\quad$ $dS \sim (dt)^{0.8}$



$H = 0.2$, $p = \frac{1}{H} = 5$ $\quad$ $H = 0.8$, $p = \frac{1}{H} = 1.25$

The usual measure of variance is

$$\sigma^2 := \frac{1}{T} 2\text{var} = \frac{1}{T} \sum_{n=0}^{N-1} |dS_n|^2.$$

If $H = 0.2$, then $dS \approx C(dt)^{0.2}$, and

$$\sigma^2 = \frac{1}{T} \sum_{n=0}^{N-1} |C(dt)^{0.2}|^2$$

$$= \frac{1}{T} N |C(\frac{T}{N})^{0.2}|^2 = T^{-0.6} C^2 N^{0.6},$$

which tends to infinity as $N \to \infty$.

If $H = 0.8$, then $dS \approx C(dt)^{0.8}$, and

$$\sigma^2 = \frac{1}{T} \sum_{n=0}^{N-1} |C(dt)^{0.8}|^2$$

$$= \frac{1}{T} N |C(\frac{T}{N})^{0.8}|^2 = T^{0.6} C^2 N^{-0.6},$$

which tends to zero as $N \to \infty$.

In theory, the variation exponent p is the value for which

$$\sum_{n=0}^{N-1} |dS_n|^p$$

has a stable value, not tending to zero or infinity as the time period dt is made smaller and smaller.

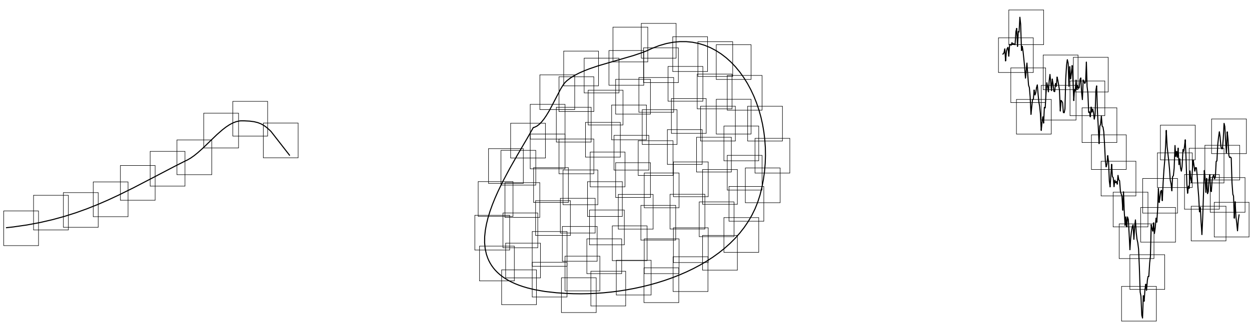
In practice (because we cannot make dt smaller and smaller) the variation exponent is not well defined.

The Itô model, $H = 0.5$ and $p = 2$, usually fits well enough. Mandelbrot has argued, however, that many price series are slightly less jagged—say $H = 0.57$ and $p = 1.75$.

Born in France in 1924, Benoit Mandelbrot spent most of his career at IBM research. He is credited with popularizing fractals in science and among the public. He is an outspoken critic of standard assumptions in finance.

See his *Fractals and Scaling in Finance: Discontinuity, Concentration, Risk*, Springer, 1997.

Mandelbrot's box dimension (just for fun)



The box dimension of an object in the plane is the power to which we should raise $1/dt$ to get the number, up to the order of magnitude, of $dt \times dt$ boxes required to cover it.

Sometimes box dimension is called *fractal dimension*. But is only one of many notions of fractal dimension.

- Object of area A: About $A/(dt)^2$ boxes are required. Box dimension = 2.

- Smooth curve of length T, about T/dt boxes are required. Box dimension = 1.

- Graph of function on $[0, T]$ with Hölder exponent H: We must cover a vertical distance $(dt)^H$ above the typical increment dt on the horizontal axis, which requires $(dt)^H/dt$ boxes. So the number of boxes needed for all T/dt increments is $T(dt)^{H-2}$. Box dimension = $2 - H$.

So the graph of an Itô process has box dimension 1.5.

2. Why does the $\sqrt{dt}$ effect happen?

Answer: Because otherwise a speculator could make a lot of money without risking bankruptcy.

- If prices are more jagged than $\sqrt{dt}$ (daily changes tend to exceed 6% of annual changes), then a simple contrarian strategy can make a lot of money.

- If prices are less jagged than $\sqrt{dt}$ (daily changes tend to be less than 6% of annual changes), then a simple momentum strategy can make a lot of money.

The $\sqrt{dt}$ effect is a consequence of market efficiency. We get $\sqrt{dt}$ when the market blocks speculators who play momentum and contrarian strategies.

Remember what the $\sqrt{dt}$ effect means in practice:

- More jagged than $\sqrt{dt}$ means $\sum_n |dS_n|^2$ is large relative to $\max_n |S_n - S_0|$.

- Less jagged than $\sqrt{dt}$ means $\sum_n |dS_n|^2$ is small relative to $\max_n |S_n - S_0|$.

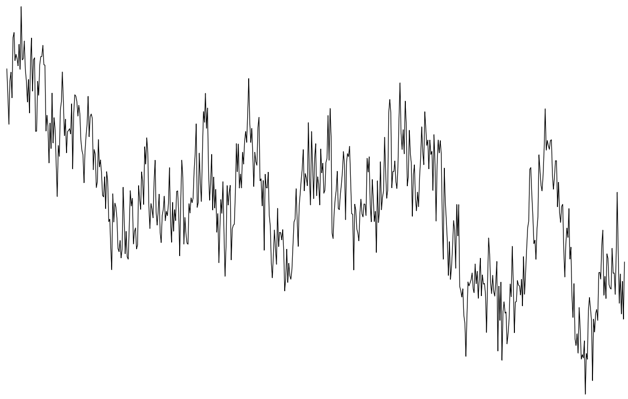
We make our claims precise this way:

- If we can count on $\sum_n |dS_n|^2 \geq \sigma^2_{\text{min}}$ and $\max_n |S_n - S_0| \leq D$, then a simple contrarian strategy can turn \$1 into \$$\sigma^2_{\text{min}}/D^2$ or more for sure.

- If we can count on $\sum_n |dS_n|^2 \leq \sigma^2_{\text{max}}$ and $\max_n |S_n - S_0| \geq D$, then a simple momentum strategy can turn \$1 into \$$D^2/\sigma^2_{\text{max}}$ or more for sure.

Too jagged

$$dS \sim (dt)^{0.2}$$



Recall that if we trade 600 times in a year, then $|dS|$ per trading period will average $(1/600)^{0.2} \approx 28\%$ of $|dS|$ over the year.

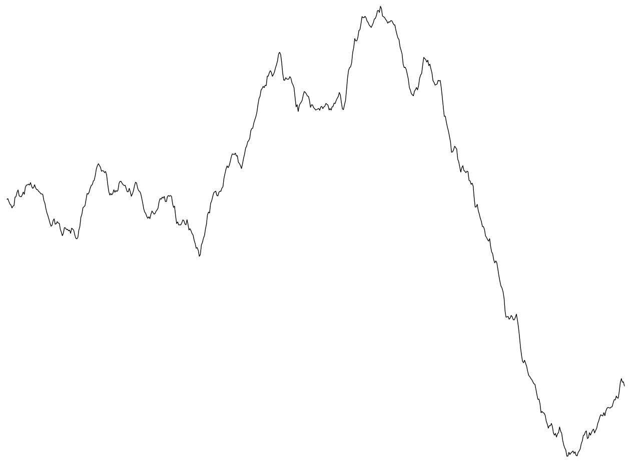
If we can count on $\sum_n |dS_n|^2 \geq \sigma^2_{\text{min}}$ and $\max_n |S_n - S_0| \leq D$, then a simple contrarian strategy can turn \$1 into $\$\sigma^2_{\text{min}}/D^2$ or more for sure.

Say we expect $10 \leq \max_n |S_n - S_0| \leq 20$. Then we might expect trading-period $|dS|$ to average more than 28% of 10, or 2.8, so that $\sum_n |dS_n|^2 \geq 600(2.8)^2 \approx 4700$.

So our contrarian strategy should turn \$1 into about $\$4700/(20)^2 \approx \$12$.

Not jagged enough

$$dS \sim (dt)^{0.8}$$



Recall that if we trade 600 times in a year, then $|dS|$ per trading period will average $(1/600)^{0.8} \approx 0.6\%$ of $|dS|$ over the year.

If we can count on $\sum_n |dS_n|^2 \le \sigma^2_{\text{max}}$ and $\max_n |S_n - S_0| \ge D$, then a simple momentum strategy can turn \$1 into $\$D^2/\sigma^2_{\text{max}}$ or more for sure.

Say we expect $10 \le \max_n |S_n - S_0| \le 20$. Then we might expect trading-period $|dS|$ to average less than 0.6% of 20, or 0.12, so that $\sum_n |dS_n|^2 \le 600(0.12)^2 \approx 9$.

So our contrarian strategy should turn \$1 into about $\$10^2/9 \approx \$11$.

The claims we want to prove:

- If we can count on $\sum_n |dS_n|^2 \geq \sigma^2_{\text{min}}$ and $\max_n |S_n - S_0| \leq D$, then a simple contrarian strategy can turn \$1 into $\$\sigma^2_{\text{min}}/D^2$ or more for sure.

- If we can count on $\sum_n |dS_n|^2 \leq \sigma^2_{\text{max}}$ and $\max_n |S_n - S_0| \geq D$, then a simple momentum strategy can turn \$1 into $\$D^2/\sigma^2_{\text{max}}$ or more for sure.

We can prove these claims rigorously in a finitary game-theoretic framework, without using any asymptotics or probability.

The Finitary Game-Theoretic Framework

Players: Investor, Market

Protocol:

$S_0 := 0$.

$\mathcal{I}_0 := 1$.

FOR $n = 1, 2, \ldots, N$:

Investor announces $M_n \in \mathbb{R}$.

Market announces $S_n \in \mathbb{R}$.

$\mathcal{I}_n := \mathcal{I}_{n-1} + M_n(S_n - S_{n-1})$.

- $S_0, \ldots, S_N$ is the price process.
- $\mathcal{I}_0, \ldots, \mathcal{I}_N$ is Investor's capital process.
- We assume perfect information; the players move in sequence and everyone sees the moves.

Apology: In most markets, prices are nonnegative. But here, for algebraic simplicity, we follow Bachelier (ordinary Brownian motion) by setting S_0 equal to zero and allowing later S_n to be negative. For the case where the S_n are constrained to be nonnegative, in the spirit of Black-Scholes (or geometric Brownian motion), see our Working Paper #5.

WHEN S_n IS NOT JAGGED ENOUGH...

Players: Investor, Market

Protocol:

$S_0 := 0$.

$\mathcal{I}_0 := 1$.

FOR $n = 1, 2, \dots, N$:

Investor announces $M_n \in \mathbb{R}$.

Market announces $S_n \in \mathbb{R}$.

$\mathcal{I}_n := \mathcal{I}_{n-1} + M_n(S_n - S_{n-1})$.

Theorem. Suppose Market is constrained to satisfy

- $\sum_n |dS_n|^2 \le \sigma^2_{\max}$ and
- $\max_n |S_n| \ge D$

for constants $\sigma^2_{\max} > 0$ and $D > 0$ known at the outset of the game. Then Investor has a strategy that guarantees

- $\mathcal{I}_n \ge 0$ for $n = 1, \dots, N$ and
- $\mathcal{I}_N \ge \frac{D^2}{\sigma^2_{\max}}$.

Lemma. If Investor plays the strategy

$$M_n = 2CS_{n-1},$$

where C is a constant, then

$$\mathcal{I}_n - 1 = C\left(S_n^2 - \sum_{i=0}^{n-1}(dS_i)^2\right). \qquad (1)$$

Proof.

$$\begin{aligned}\mathcal{I}_n - \mathcal{I}_{n-1} &= 2CS_{n-1}\left(S_n - S_{n-1}\right)\\ &= C\left(S_n^2 - S_{n-1}^2 - (S_n - S_{n-1})^2\right).\end{aligned}$$

Proof of theorem. Consider the momentum strategy

$$M_n = 2\frac{1}{\sigma_{\max}^2}S_{n-1}.$$

By Eq. (1),

$$\mathcal{I}_n - 1 = \frac{1}{\sigma_{\max}^2}S_n^2 - \frac{1}{\sigma_{\max}^2}\sum_{i=0}^{n-1}(dS_i)^2 \geq \frac{1}{\sigma_{\max}^2}S_n^2 - 1.$$

Hence

$$\mathcal{I}_n \geq \frac{1}{\sigma_{\max}^2}S_n^2 \geq 0$$

for all n. Let n^* be the first n for which $|S_n| \geq D$. If we stop play at n^* (i.e., play $M_n = 0$ for $n \geq n^*$), we obtain

$$\mathcal{I}_N = \mathcal{I}_{n^*} \geq \frac{D^2}{\sigma_{\max}^2}.$$

WHEN S_n IS TOO JAGGED...

Players: Investor, Market

Protocol:

$S_0 := 0.$

$\mathcal{I}_0 := 1.$

FOR $n = 1, 2, \ldots, N$:

Investor announces $M_n \in \mathbb{R}$.

Market announces $S_n \in \mathbb{R}$.

$\mathcal{I}_n := \mathcal{I}_{n-1} + M_n(S_n - S_{n-1})$.

Theorem. Suppose Market is constrained to satisfy

- $\sum_n |dS_n|^2 \geq \sigma^2_{\mathrm{min}}$ and
- $\max_n |S_n| \leq D$

for constants $\sigma^2_{\mathrm{min}} > 0$ and $D > 0$ known at the outset of the game. Then Investor has a strategy that guarantees

- $\mathcal{I}_n \geq 0$ for $n = 1, \ldots, N$ and
- $\mathcal{I}_N \geq \frac{\sigma^2_{\mathrm{min}}}{D^2}$.

The Same Lemma. If Investor plays the strategy

$$M_n = 2CS_{n-1},$$

where C is a constant, then

$$\mathcal{I}_n - 1 = C\left(S_n^2 - \sum_{i=0}^{n-1}(dS_i)^2\right).$$

Proof of theorem. Consider the contrarian strategy

$$M_n = -2\frac{1}{D^2}S_{n-1}.$$

By the lemma,

$$\mathcal{I}_n - 1 = \frac{1}{D^2}\sum_{i=0}^{n-1}(dS_i)^2 - \frac{1}{D^2}S_n^2 \geq \frac{1}{D^2}\sum_{i=0}^{n-1}(dS_i)^2 - 1.$$

Hence

$$\mathcal{I}_n \geq \frac{1}{D^2}\sum_{i=0}^{n-1}(dS_i)^2 \geq 0$$

for all n, and

$$\mathcal{I}_N \geq \frac{1}{D^2}\sum_{i=0}^{N-1}(dS_i)^2 \geq \frac{\sigma_{\text{min}}^2}{D^2}.$$

3. Making the picture infinitary using an ultraproduct of games

Our finitary theory is fully rigorous, and it is more meaningful than an infinitary theory that pretends trading can take place infinitely often.

But infinitary theory can simplify the picture, so that we can see and remember essential features more easily.

- From a practical point of view, the $\sqrt{dt}$ effect says merely that $\sum_n (dS_n)^2$ is stable over a practical range of trading frequencies, perhaps from $dt = 1$ day to $dt = 10$ years.

- The infinitary picture gives us a distorted but sharpened view. Through its prism, we see $\sum_n (dS_n)^2$ becoming infinitely large or small when $p \neq 2$, opening the prospect for fantastic arbitrage profits.

The Infinitary Picture

Divide $[0, T]$ into an infinite number N of trading periods, each of infinitesimal length $dt = T/N$. This takes us outside ordinary analysis, towards nonstandard analysis.

But we do not need much nonstandard analysis. The notion of an ultraproduct is enough.

An ultraproduct is an infinite Cartesian product, shrunk back down by taking equivalence classes with respect to an ultrafilter.

- The hyperreal numbers, which include infinite and infinitesimal numbers along with real numbers, are one example. In this example, the ultraproduct is an ultrapower: the product of an infinite number of copies of the same space (the real numbers).
- **Our infinitary game is an ultraproduct of finitary games, with finite but successively larger numbers of trading periods.**

Jerzy Łoś invented ultraproducts in the 1950s.

In the 1970s, Abraham Robinson promoted nonstandard analysis as a tool for proving results in standard analysis, which already involves infinitary idealizations.

Our ultraproduct construction, in contrast, stands on its own. It is something standard analysis cannot provide: an infinitary picture of a game.

Our exposition of ultraproducts in *Probability and Finance: It's Only a Game* needs improvement. The exposition in our Working Paper #5 may be better.

References on Ultraproducts

- Łoś's seminal article is "*Quelques remarques, théorems et problèmes sur les classes définissables d'algèbres*", in *Mathematical Interpretation of Formal Systems*, edited by Th. Skolem, et al., North-Holland 1955.

- See also J. L. Bell and A. B. Slomson's *Models and Ultraproducts: An Introduction*, North-Holland 1969,1972.

- A good book on nonstandard analysis: Robert Goldblatt's *Lectures on the Hyperreals: An Introduction to Nonstandard Analysis*, Springer 1998.

- See also my Spring 2002 public lectures, at http://www.glennshafer.com/courses.html.

Start with an ultrafilter.

An *ultrafilter* in the set $\mathbb{N}$ of natural numbers is a family F of subsets of $\mathbb{N}$ such that

1. $\mathbb{N} \in F$,

2. if $A \in F$ and $A \subseteq B \subseteq \mathbb{N}$, then $B \in F$,

3. if $A \in F$ and $B \in F$, then $A \cap B \in F$,

4. if $A \subseteq \mathbb{N}$, then either $A \in F$ or $\mathbb{N} \setminus A \in F$, and

5. $\emptyset \notin F$ (so by 3, not both A and $\mathbb{N} \setminus A$ are in F).

An ultrafilter F is *nontrivial* if it does not contain a set consisting of a single integer. This implies that all the sets in F are infinite.

By the axiom of choice, a nontrivial ultrafilter exists. Fix one and call it F.

Nonconstructive: we do not construct F.

But now we construct the hyperreals from F.

1. $\mathbb{N} \in F$,
2. if $A \in F$ and $A \subseteq B \subseteq \mathbb{N}$, then $B \in F$,
3. if $A \in F$ and $B \in F$, then $A \cap B \in F$,
4. if $A \subseteq \mathbb{N}$, then either $A \in F$ or $\mathbb{N} \setminus A \in F$, and
5. $\emptyset \notin F$ (so by 3, not both A and $\mathbb{N} \setminus A$ are in F).

A property of natural numbers holds for *most* natural numbers (*for most* k) if the set for which it holds is in F.

- 2: If A holds for most k and A implies B, then B holds for most k.
- 3: If A holds for most k and B holds for most k, then $A \cap B$ holds for most k.
- 4: Either A or its negation holds for most k.
- 5: Not both A and its negation hold for most k.

Our (arbitrarily chosen) ultrafilter amounts to an arbitrary rule for deciding which of two or more infinite sets of numbers has a majority.

- Partition $\mathbb{N}$ into two sets A_1 and A_2. One of the two has the majority (is in F).
- Partition $\mathbb{N}$ into a finite number of sets $A_1, \ldots, A_n$. Exactly one of the A_i has the majority (is in F).

Ultrapower construction of the hyperreals

A *hyperreal number* is a sequence $\left[a^{(1)}a^{(2)}\ldots\right]$ of real numbers.

Define operations term by term:

$$\left[a^{(1)}a^{(2)}\ldots\right]+\left[b^{(1)}b^{(2)}\ldots\right]:=[\left(a^{(1)}+b^{(1)}\right)\left(a^{(2)}+b^{(2)}\right)\ldots]$$

Define relations by voting:

$$[a^{(1)}a^{(2)}\ldots]\leq[b^{(1)}b^{(2)}\ldots] \text{ if } a^{(k)}\leq b^{(k)} \text{ for most } k$$

If a and b are hyperreals, then

$$a<b, a=b, \text{ or } a>b.$$

One and only one of the three conditions holds.

We identify hyperreals a and b such that $a=b$. So a hyperreal is really an equivalence class: $[a^{(1)}a^{(2)}\ldots]$ is the equivalence class containing $a^{(1)}a^{(2)}\ldots$.

- Embed the reals in the hyperreals by identifying each real a with $[a, a, \ldots]$.

- A hyperreal a is *finite* if there are reals B and C such that $B < a < C$.

- A sequence of reals that increases without bound (e.g., $1, 2, 4, \ldots$) defines an infinite hyperreal (e.g., $[1, 2, 4, \ldots]$). The faster the sequence grows, the larger the hyperreal.

- A hyperreal a is *infinitesimal* if $|a| < \frac{1}{C}$ for every natural number C.

These ideas lead to a rigorous definition of q-variation for any function S on $[0, T]$.

For $N = 1, 2, \ldots$,

1. divide $[0, T]$ into N periods of length $dt = \frac{T}{N}$,
2. observe $S_0, S_1, \ldots, S_N$, where $S_n = S(ndt)$,
3. set $dS_n := S_{n+1} - S_n$, and
4. set
$$a^{(N)} := \sum_{n=0}^{N-1} |dS_n|^q.$$

Set qvar $:= [a^{(1)}, a^{(2)}, \ldots]$.

The *variation exponent* is the unique real number p such that qvar is infinitely large for $q < p$ and infinitesimal for $q > p$.

This construction is an example of an ultra-product. For each number N, we have a finitary structure. We are looking at the ultra-product of these finitary structures.

But this construction is only a trial run.

We really want an ultraproduct of games.

Do not begin with S defined on $[0, T]$.

Begin instead with a sequence of games, one for each integer N. In the game labeled N...

- Investor makes moves $S_0, \ldots, S_N$.
- This determines $\sum_{n=0}^{N-1} |dS_n|^q$ for all $q \geq 0$.

These numbers define the hyperreals qvar and thence the variation exponent p.

This ultraproduct of games provides an infinitary framework in which we can derive the $\sqrt{dt}$ effect elegantly—without messy constants D and σ^2_{max} or σ^2_{min}.

For $N = 1, 2, \ldots$, play a market game. . .

Protocol:

$S_0 := 0$.

$\mathcal{I}_0 := 1$.

FOR $n = 1, 2, \ldots, N$:

Investor announces $M_n \in \mathbb{R}$.

Market announces $S_n \in \mathbb{R}$.

$\mathcal{I}_n := \mathcal{I}_{n-1} + M_n(S_n - S_{n-1})$.

$$\mathsf{maxjump}^{(N)} := \max_{n=0,\ldots,N-1} |dS_n|$$

$$\mathsf{qvar}^{(N)} := \sum_{n=0}^{N-1} |dS_n|^q$$

$$\mathsf{maxdeviation}^{(N)} := \max_{n=1,\ldots,N} |S_n|$$

$$\mathsf{finalcap}^{(N)} := \mathcal{I}_N$$

Choose an ultrafilter in $\mathbb{N}$ and form the ultraproduct. When Investor and Market have played the individual games for all N, they have played this ultraproduct game.

The biggest price jump in one period in the ultraproduct game is

$$\mathsf{maxjump} := [\mathsf{maxjump}^{(1)}, \mathsf{maxjump}^{(2)}, \ldots].$$

Since the price starts at 0, the largest deviation from the initial price in the course of the ultraproduct game is

$$\mathsf{maxdeviation} := [\mathsf{maxdeviation}^{(1)}, \mathsf{maxdeviation}^{(2)}, \ldots].$$

We impose two constraints on Market, which tie together how he plays in the different games:

- Market must keep $\mathsf{maxjump} \leq \epsilon$, where ϵ is a positive infinitesimal number announced in advance. (This is a continuity condition.)
- Market cannot make $\mathsf{maxdeviation}$ infinite or infinitesimal.

The q-variation of Market's play in the ultraproduct game is

$$\mathsf{qvar} := [\mathsf{qvar}^{(1)}, \mathsf{qvar}^{(2)}, \ldots].$$

The variation exponent p for Market's play in the ultraproduct game is the point where qvar goes from being infinite to being infinitesimal.

Investor's final capital in the ultraproduct game is

$$\mathsf{finalcap} := [\mathsf{finalcap}^{(1)}, \mathsf{finalcap}^{(2)}, \ldots].$$

Theorem. If Market plays so as to avoid allowing Investor to become infinitely rich by playing his simple momentum or contrarian strategies, then the variation exponent will come out equal to 2.

4. Making the picture infinitary using measure-theoretic probability

The study of 2-variation for diffusion processes dates back at least to Paul Lévy's work in the 1930s. Dominique Lepingle studied p-variation for semi-martingales in the 1970s.

The fact that simple momentum and contrarian strategies force $p = 2$ seems to have first emerged in the late 1980s, in the context of fractional Brownian motion. (Rogers, who published the result in 1997, claims to have been aware of it in 1989.)

A fractional Brownian motion with variation exponent not equal to 2 has long-range autocorrelation. Because autocorrelation is probability-ese for predictability, it is not surprising that an autocorrelated price process provides arbitrage opportunities.

References

- La variation d'ordre p des semi-martingales, by Dominique Lepingle, *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 36, 295–316, 1976.

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- Arbitrage with fractional Brownian motion, by L. Chris G. Rogers, *Mathematical Finance*, 7, 95–105, 1997.

- Arbitrage in fractional Brownian motion models, by Patrick Cheridito, *Finance and Stochastics*, 7:4, 533-553, October, 2003.

- For other references, see §11.6 of *Probability and Finance: It's Only a Game!*

5. Advantages of the ultraproduct picture over the measure-theoretic picture

1. It is simpler.

2. The measure-theoretic picture lends itself to overinterpretation at two levels:

   - As analysts, we can take too seriously the fiction of a function $S : [0, T] \mapsto \mathbb{R}$ lurking behind the observed $S_0, \ldots, S_N$.

   - As probabilists, we can take too seriously the fiction of an exact probability measure lurking behind the market.

3. The ultraproduct is a more transparent simplification. It adds no extra elements like probability to the picture. It keeps us aware that the market is only a game.

An example of measure-theoretic confusion.

As mentioned earlier, some market data suggest a variation exponent slightly less than 2; perhaps $p \approx 1.75$.

Mandelbrot has proposed various probability models to explain this.

But the best explanation is to be found in a realistic finitary game, not in an infinitary idealization.

Price changes can be driven by new information or by speculation. If there is little new information, then speculation is mainly responsible for pushing p up from 1 (its value when there is no variation). Market frictions (transaction costs and limited trading frequency) can keep p from being pushed all the way to 2.