Introduction to Game-Theoretic Probability

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January 28, 2002

- The project: Replace measure theory with game theory.
- The game-theoretic strong law.
- Game-theoretic price and probability.
- The game-theoretic central limit theorem
- The game-theoretic Black-Scholes formula

THE PROJECT: Replace game theory with measure theory as a framework for probability and finance

- Classical theorems in probability become theorems about games where a player may bet on certain specified events at specified odds but *no stochasticity is assumed*.

- No stochastic assumption is needed for option pricing.

- CAPM can be derived with no assumptions of stochasticity and no assumptions about beliefs and preferences of investors.

Probability and Finance: It's Only a Game!
Glenn Shafer and Volodya Vovk, Wiley 2001
http://www.cs.rhul.ac.uk/home/vovk/book/

THE PROJECT: Replace game theory with measure theory

MEASURE-THEORETIC FRAMEWORK

- Start with prices for everything.
- Basic framework (measure space) is static. Filtration is added to model time.
- Draw conclusions "except for a set of measure zero" or "with high probability".

GAME-THEORETIC FRAMEWORK

- Limited prices (betting offers).
- Sequential perfect-information game.
- Prices may be given at the outset. Or they may be set in the course of the game!!
- Lower and upper prices can be derived for all payoffs.
- Draw conclusions with high lower probability.

The classical limit theorems (law of large numbers, law of iterated logarithm, central limit theorem) are theorems about a two-player perfect-information game.

On each round of the game:

Player I	(Skeptic)	bets on what Reality will do.
Player II	(Reality)	decides what to do.

Each theorem says that Skeptic has a winning strategy when he is set a certain goal.

Example: Coin Tossing

On each round, Skeptic bets as much as he wants on heads or tails, at even odds. Skeptic wins if (1) he does not go broke, and (2) either he becomes infinitely rich or else the proportion of heads converges to one-half.

Theorem: Skeptic has a winning strategy.

THE STRONG LAW OF LARGE NUMBERS FOR COIN TOSSING

Players: Skeptic, Reality

Protocol:

$\mathcal{K}_0 = 1$.

FOR $n = 1, 2, \ldots$:

Skeptic announces $M_n \in \mathbb{R}$.

Reality announces $x_n \in \{-1, 1\}$.

$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n$.

Winner:

Skeptic wins if

(1) $\mathcal{K}_n$ is never negative and

(2) either $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} x_i = 0$

or else $\lim_{n\to\infty} \mathcal{K}_n = \infty$.

Otherwise Reality wins.

PROPOSITION:

Skeptic has a winning strategy.

Generalize by letting Skeptic choose any number in the interval $[-1, 1]$. Then we get a strong law of large numbers for a bounded sequence of variables $x_1, x_2, \ldots$. (Don't call them "random variables", because they have no probability distribution—just a price of zero on each round.)

> **Players:** Skeptic, Reality
> **Protocol:**
> $\mathcal{K}_0 = 1$.
> FOR $n = 1, 2, \ldots$:
> Skeptic announces $M_n \in \mathbb{R}$.
> Reality announces $x_n \in [-1, 1]$.
> $\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n$.
> **Winner:**
> Skeptic wins if
> (1) $\mathcal{K}_n$ is never negative and
> (2) either $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i = 0$
> or else $\lim_{n\to\infty} \mathcal{K}_n = \infty$.
> Otherwise Reality wins.

PROPOSITION:
Skeptic has a winning strategy.

Generalize further by letting another player (allied with Reality) set the prices on each round.

Players: Skeptic, Reality
Protocol:
$\mathcal{K}_0 = 1$.
FOR $n = 1, 2, \ldots$:
Forecaster announces $m_n \in \mathbb{R}$.
Skeptic announces $M_n \in \mathbb{R}$.
Reality announces $x_n \in [m_n - 1, m_n + 1]$.
$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n(x_n - m_n)$.
Winner:
Skeptic wins if
(1) $\mathcal{K}_n$ is never negative and
(2) either $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}(x_i - m_i) = 0$
or else $\lim_{n\to\infty} \mathcal{K}_n = \infty$.
Otherwise Reality wins.

PROPOSITION:
Skeptic has a winning strategy.

PRICE AND PROBABILITY

> $\mathcal{K}_0 := \alpha$.
> FOR $n = 1, \ldots, N$:
>   Skeptic announces $M_n \in \mathbb{R}$.
>   Reality announces $x_n \in \{-1, 1\}$.
> $\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n$.

Upper Price for a Variable y:

$\overline{\mathbb{E}}\, y :=$ smallest initial stake Skeptic can parlay into y or more at the end of the game

$$= \inf\{\mathcal{L}(\Box) | \mathcal{L} \text{ is a martingale and } \mathcal{L}(x_1, \ldots, x_N) \geq y(x_1, \ldots, x_N)\}.$$

A *martingale* is a capital process for Skeptic.

Suppose Skeptic is willing to sell a variable to the public at any price at which he can replicate it with no risk of loss. Then $\overline{\mathbb{E}}\, y$ is his *minimum selling price* for y.

Upper Price for a Variable y:

$$\overline{\mathbb{E}}\, y := \text{ smallest initial stake Skeptic can parlay into } y \text{ or more at the end of the game}$$
$$= \text{ Skeptic's minimum selling price for } y.$$

Buying y for α is the same as selling $-y$ for $-\alpha$. So $-\overline{\mathbb{E}}\, -y$ is Skeptic's maximum buying price for y. We call this its lower price:

$$\underline{\mathbb{E}}\, y := -\overline{\mathbb{E}}\, -y.$$

Probability from Price

$$\overline{\mathbb{P}}\, E := \overline{\mathbb{E}}\, \mathcal{I}_E \qquad \text{and} \qquad \underline{\mathbb{P}}\, E := \underline{\mathbb{E}}\, \mathcal{I}_E,$$

where $\mathcal{I}_E$ is the indicator variable for E.

$$\overline{\mathbb{P}}\, E := \overline{\mathbb{E}}\, \mathcal{I}_E = \text{ smallest initial stake Skeptic can parlay into at least 1 if } E \text{ happens and at least 0 otherwise}$$

THE CENTRAL LIMIT THEOREM

We consider only coin-tossing (DeMoivre's theorem). For simplicity, we now score Heads as $1/\sqrt{N}$ and Tails as $-1/\sqrt{N}$.

FOR $n = 1, \ldots, N$:
  Skeptic announces $M_n \in \mathbb{R}$.
  Reality announces $x_n \in \{-\frac{1}{\sqrt{N}}, \frac{1}{\sqrt{N}}\}$.
  $\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n$.

Set $\mathcal{S}_n := \sum_{i=1}^{n} x_i$.

Consider a smooth function U.

De Moivre's Theorem For N sufficiently large, both $\overline{\mathbb{E}}\, U(\mathcal{S}_N)$ and $\underline{\mathbb{E}}\, U(\mathcal{S}_N)$ are arbitrarily close to $\int_{-\infty}^{\infty} U(z)\, \mathcal{N}_{0,1}(dz)$.

How do we prove De Moivre's theorem?

$$\mathcal{S}_n := \sum_{i=1}^{n} x_i.$$

We want to know the price at time 0 of the payoff $U(\mathcal{S}_N)$ at time N. Let us also consider its price at time n. Intuitively, this should depend on $\mathcal{S}_n$, the value of the sum so far. Assume, optimistically, that the price at time n is given by a function of two variables, $\overline{U}(s, D)$: the price at time n is $\overline{U}(\mathcal{S}_n, \frac{N-n}{N})$.

Successive prices are

$$\overline{U}(0,1), \overline{U}(\mathcal{S}_1, \tfrac{N-1}{N}), \ldots$$
$$\ldots, \overline{U}(\mathcal{S}_{N-1}, \tfrac{1}{N}), \overline{U}(\mathcal{S}_N, 0),$$

These must be the successive values of a martingale.

- $\overline{U}(\mathcal{S}_N, 0)$ must equal $U(\mathcal{S}_N)$.

- $\overline{U}(0,1)$ is the price that interests us.

We want to choose $\overline{U}(s, D)$ so that

$$\overline{U}(0,1), \overline{U}(\mathcal{S}_1, \tfrac{N-1}{N}), \ldots$$
$$\ldots, \overline{U}(\mathcal{S}_{N-1}, \tfrac{1}{N}), \overline{U}(\mathcal{S}_N, 0)$$

is a martingale with $\overline{U}(\mathcal{S}_N, 0) = U(\mathcal{S}_N)$.

Consider the increments in s, D, and $\overline{U}$:

- $\Delta s_n = x_n = \pm\frac{1}{\sqrt{N}}$.
- $\Delta D_n = -\frac{1}{N}$.
- $\Delta \overline{U}_n = \overline{U}(\mathcal{S}_n, \frac{N-n}{N}) - \overline{U}(\mathcal{S}_{n-1}, \frac{N-n+1}{N})$.

Study $\Delta \overline{U}$ with a Taylor's expansion:

$$\Delta \overline{U} \approx \frac{\partial \overline{U}}{\partial s}\Delta s + \frac{\partial \overline{U}}{\partial D}\Delta D + \frac{1}{2}\frac{\partial^2 \overline{U}}{\partial s^2}(\Delta s)^2$$

$$= \frac{\partial \overline{U}}{\partial s}x - (\frac{\partial \overline{U}}{\partial D} - \frac{1}{2}\frac{\partial^2 \overline{U}}{\partial s^2})\frac{1}{N}.$$

$$\Delta\overline{U} \approx \frac{\partial\overline{U}}{\partial s}x - (\frac{\partial\overline{U}}{\partial D} - \frac{1}{2}\frac{\partial^2\overline{U}}{\partial s^2})\frac{1}{N}.$$

We need the second term to go away, which requires

$$\frac{\partial\overline{U}}{\partial D} = \frac{1}{2}\frac{\partial^2\overline{U}}{\partial s^2}$$

Then we obtain the desired martingale by buying $\frac{\partial\overline{U}}{\partial s}$ x-tickets on the nth round. In other words, we set

$$M_n := \frac{\partial\overline{U}}{\partial s}(\mathcal{S}_{n-1}, \frac{N-n+1}{N}).$$

The partial differential equation

$$\frac{\partial\overline{U}}{\partial D} = \frac{1}{2}\frac{\partial^2\overline{U}}{\partial s^2}$$

is the *heat equation*. Laplace showed that its solution is a Gaussian integral.

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With the initial condition $\overline{U}(s,0) = U(s)$, the solution is

$$\begin{aligned}\overline{U}(s,D) &= \int_{-\infty}^{\infty} U(z)\,\mathcal{N}_{s,D}(dz)\\ &= \int_{-\infty}^{\infty} U(s+z)\,\mathcal{N}_{0,D}(dz).\end{aligned}$$

So the initial price, $\overline{U}(0,1)$, is

$$\int_{-\infty}^{\infty} U(z)\,\mathcal{N}_{0,1}(dz).$$

THE BLACK-SCHOLES PROTOCOL

The price of a security $\mathcal{S}$ is determined by a game just like those we have been studying. If we write $S(t)$ for the price at time t, then we can write the game's protocol as follows.

Parameters: $T > 0$ and $N \in \mathbb{N}$; $dt := T/N$
Players: Investor, Market
Protocol:
  $\mathcal{I}(0) := 0$.
  Market announces $S(0) > 0$.
  FOR $t = 0, dt, 2dt, \ldots, T - dt$:
    Investor announces $\delta(t) \in \mathbb{R}$.
    Market announces $dS(t) \in \mathbb{R}$.
    $S(t + dt) := S(t) + dS(t)$.
    $\mathcal{I}(t + dt) := \mathcal{I}(t) + \delta(t)dS(t)$.

A *European option* on a stock $\mathcal{S}$ with maturity T is a security that pays the amount $U(S(T))$ at time T, where U is a known function. If $\overline{\mathbb{E}}\, U(S(T)) = \underline{\mathbb{E}}\, U(S(T))$, then we say that *the option is priced*.

STOCHASTIC BLACK-SCHOLES

Parameters: $T > 0$ and $N \in \mathbb{N}$; $dt := T/N$

Players: Investor, Market

Protocol:

$\mathcal{I}(0) := 0$.

Market announces $S(0) > 0$.

FOR $t = 0, dt, 2dt, \ldots, T - dt$:

  Investor announces $\delta(t) \in \mathbb{R}$.

  Market announces $dS(t) \in \mathbb{R}$.

  $S(t + dt) := S(t) + dS(t)$.

  $\mathcal{I}(t + dt) := \mathcal{I}(t) + \delta(t)dS(t)$.

Constraint on Market: Market must choose $dS(t)$ randomly: $dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$, where $W(t)$ is a standard Brownian motion.

With this constraint on Market, $U(S(T))$ is priced:

$$\begin{aligned}\overline{\mathbb{E}}\,U(S(T)) &= \underline{\mathbb{E}}\,U(S(T)) \\ &= \int_{-\infty}^{\infty} U\left(S(0)e^{z}\right) \mathcal{N}_{-\sigma^2 T/2,\,\sigma^2 T}(dz).\end{aligned}$$

- **Textbook Stochastic Black-Scholes:** Security $\mathcal{S}$ is priced by the market. Its price $S(t)$ is assumed to follow geometric Brownian motion; σ^2 can be estimated from past $dS(t)$. All options are priced by plugging the estimate of σ^2 into the Black-Scholes formula.

- **Stochastic Black-Scholes in Practice:** Security $\mathcal{S}$ is priced by the market. Puts/calls on $\mathcal{S}$ are also priced by the market. (These form a two-dimensional array: a range of strikes and a range of maturities.) Inconsistencies in the put/call prices of show that the assumption of geometric Brownian motion for $S(t)$ is faulty (volatility smile). So ad hoc adjustments are required to price other options.

- **Vovk's Game-Theoretic Black-Scholes:** Instead of having a market price puts/calls, have it price a dividend-paying security $\mathcal{D}$. Each day until maturity, $\mathcal{D}$ pays the dividend $(dS(t)/S(t))^2$. Now we need only a one-dimensional array: one $\mathcal{D}$ for each maturity. All other options on $\mathcal{S}$ with that maturity are priced by plugging the market price of $\mathcal{D}$ into the Black-Scholes formula. No stochastic assumptions or ad hoc adjustments are required.

Purely Game-Theoretic Black-Scholes

Investor trades in two securities: $\mathcal{S}$, which pays no dividends and $\mathcal{D}$, which pays the dividend $(dS(t)/S(t))^2$.

Parameters: $T > 0$ and $N \in \mathbb{N}$; $dt := T/N$
Players: Investor, Market
Protocol:
Market announces $S(0) > 0$ and $D(0) > 0$.
$\mathcal{I}(0) := 0$.
FOR $t = 0, dt, 2dt, \ldots, T - dt$:
Investor announces $\delta(t) \in \mathbb{R}$ and $\lambda(t) \in \mathbb{R}$.
Market announces $dS(t) \in \mathbb{R}$ and $dD(t) \in \mathbb{R}$.
$S(t + dt) := S(t) + dS(t)$.
$D(t + dt) := D(t) + dD(t)$.
$$\mathcal{I}(t + dt) := \mathcal{I}(t) + \delta(t)dS(t) + \lambda(t)\left(dD(t) + (dS(t)/S(t))^2\right).$$
Constraints on Market: (1) $D(t) > 0$ for $0 < t < T$ and $D(T) = 0$, (2) $S(t) \geq 0$ for all t, and (3) the wildness of Market's moves is constrained.

Once D pays its last dividend, at time T, it is worthless: $D(T) = 0$. So Market is constrained to make his $dD(t)$ add to $-D(0)$.

$$d\mathcal{I}(t) = \delta(t)dS(t) + \lambda(t)\left(dD(t) + (dS(t)/S(t))^2\right)$$

$$d\overline{U}(S(t), D(t)) \approx \frac{\partial \overline{U}}{\partial s}dS(t) + \frac{\partial \overline{U}}{\partial D}dD(t) + \frac{1}{2}\frac{\partial^2 \overline{U}}{\partial s^2}(dS(t))^2$$

Game-theoretic Black-Scholes equation:
We need

$$\delta(t) = \frac{\partial \overline{U}}{\partial s}, \qquad \lambda(t) = \frac{\partial \overline{U}}{\partial D}, \qquad \frac{\lambda(t)}{S^2(t)} = \frac{1}{2}\frac{\partial^2 \overline{U}}{\partial s^2}.$$

The two equations involving $\lambda(t)$ require that the function $\overline{U}$ satisfy

$$-\frac{\partial \overline{U}}{\partial D} + \frac{1}{2}s^2\frac{\partial^2 \overline{U}}{\partial s^2} = 0$$

for all s and all $D > 0$.

Game-theoretic Black-Scholes formula:
With initial condition $\overline{U}(s, 0) = U(s)$, the solution is

$$\overline{U}(s, D) = \int_{-\infty}^{\infty} U\left(se^z\right)\mathcal{N}_{-D/2,D}(dz).$$

To summarize, the price at time t for the European option $\mathcal{U}$ in a market where both the underlying security $\mathcal{S}$ and a volatility security $\mathcal{D}$ with dividend $(dS(t)/S(t))^2$ are traded is

$$\mathcal{U}(t) = \int_{-\infty}^{\infty} U\left(S(t)e^{z}\right) \mathcal{N}_{-D(t)/2,D(t)}(dz).$$

To hedge this price, we hold a continuously changing portfolio, containing

$$\frac{\partial \overline{U}}{\partial s}(S(t), D(t)) \text{ shares of } \mathcal{S}$$

and

$$\frac{\partial \overline{U}}{\partial D}(S(t), D(t)) \text{ shares of } \mathcal{D}$$

at time t.

Our game-theoretic Black-Scholes theory can be contrasted with two versions of the stochastic theory:

- **Textbook Stochastic Black-Scholes:**
  Security $\mathcal{S}$ is priced by the market. Its price $S(t)$ is assumed to follow geometric Brownian motion; σ^2 can be estimated from past $dS(t)$. All options are priced by plugging the estimate of σ^2 into the Black-Scholes formula.

- **Stochastic Black-Scholes in Practice:**
  Security $\mathcal{S}$ is priced by the market. Puts/calls on $\mathcal{S}$ are also priced by the market. (These form a two-dimensional array: a range of strikes and a range of maturities.) Inconsistencies in the put/call prices of show that the assumption of geometric Brownian motion for $S(t)$ is faulty (volatility smile). So ad hoc adjustments are required to price other options.

- **Game-Theoretic Black-Scholes:**
  Security $\mathcal{S}$ is priced by the market. Dividend-paying security $\mathcal{D}$ is also priced by the market (this is only a one-dimensional array: a range of maturities). All other options on $\mathcal{S}$ are priced by plugging the market price of $\mathcal{D}$ into the Black-Scholes formula. No stochastic assumptions or ad hoc adjustments are required.

We are calling for a far-reaching change in how option exchanges are organized. The change will be hard to sell and complex to implement but should greatly increase efficiency.