The Martingale Proof of the Strong Law of Large Numbers

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- The game for the strong law.
- The strong law as a proposition about the game.
- Game-theoretic variables and processes.
- Game-theoretic martingales.
- Proof of the game-theoretic strong law.
- Game-theoretic strong law $\Rightarrow$ measure-theoretic strong law.

THE GAME FOR THE STRONG LAW

Players: Skeptic, Reality

Protocol:

$\mathcal{K}_0 = 1$.
FOR $n = 1, 2, \ldots$:
  Skeptic announces $M_n \in \mathbb{R}$.
  Reality announces $x_n \in [-1, 1]$.
  $\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n$.

Winner: Skeptic wins if (1) $\mathcal{K}_n$ is never negative and (2) either

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i = 0$$

or

$$\lim_{n\to\infty} \mathcal{K}_n = \infty.$$

Otherwise Reality wins.

THE STRONG LAW AS A PROPOSITION ABOUT THE GAME

Proposition 1 *Skeptic has a winning strategy.*

Who wins? Skeptic wins if (1) $\mathcal{K}_n$ is never negative and (2) either

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} x_i = 0 \qquad \text{or} \qquad \lim_{n\to\infty} \mathcal{K}_n = \infty.$$

Otherwise Reality wins.

So Proposition 1 can be restated this way: Skeptic has a strategy that (1) does not risk bankruptcy (does not risk $\mathcal{K}_n < 0$ for any n) and (2) guarantees that either the average of the first n of the x_i converges to 0 or else Skeptic becomes infinitely rich.

Loosely:
The average of the first n of the x_i converges to 0 unless Skeptic becomes infinitely rich.

THE IDEA OF THE PROOF

—Proof is constructive; we spell out Skeptic's winning strategy.
—The winning strategy is computable and therefore measurable.

Idea 1 Establish an account for betting on heads. On each round, bet ϵ of the account on heads. Then Reality can keep the account from getting indefinitely large only by eventually holding the cumulative proportion of heads at or below $\frac{1}{2}(1+\epsilon)$. **It does not matter how little money the account starts with.**

Idea 2 Establish infinitely many accounts. Use the kth account to bet on heads with $\epsilon = 1/k$. This forces the cumulative proportion of heads to stay at 1/2 or below.

Idea 3 Set up similar accounts for betting on tails. This forces Reality to make the proportion converge exactly to one-half.

GAME-THEORETIC VARIABLES

- A *path* is an infinite sequence $x_1x_2\dots$ of numbers satisfying $-1 \leq x_n \leq 1$. Reality's moves form a path.

- The set of all paths is called the *sample space* and designated by Ω:

$$\Omega = [-1, 1]^\infty.$$

- A function on Ω is a *variable*. The x_n can be thought of as variables.

- A subset of Ω is an *event*.

SITUATIONS

- A *situation* is a finite initial sequence of moves by Reality. Write $\Omega^\diamond$ for the set of all situations.

- Designate by $\Box$ the *initial situation*, the sequence of length zero.

- Situation s *precedes* situation t ($s \sqsubseteq t$) if t contains s. This means $s = x_1 x_2 \ldots x_m$ and $t = x_1 x_2 \ldots x_m \ldots x_n$.

- Situations s and t *diverge* if neither precedes the other.

- Write $|s|$ for s's length: $|x_1 x_2 \ldots x_n| = n$.

- When ξ is a path, say $\xi = x_1 x_2 \ldots$, we write ξ^n for the situation $x_1 x_2 \ldots x_n$.

GAME-THEORETIC PROCESSES AND MARTINGALES

- A real-valued function on $\Omega^\diamond$ is a *process*.

- Any process $\mathcal{P}$ can be interpreted as a strategy for Skeptic: interpret $\mathcal{P}(x_1 \dots x_{n-1})$ as the amount of x_n Skeptic is to buy in situation $x_1 \dots x_{n-1}$.

- A strategy for Skeptic, together with a particular initial capital for Skeptic, also defines a process: Skeptic's *capital process*.

- We also call a capital process for Skeptic a *martingale*.

NOTATION FOR MARTINGALES

Skeptic begins with capital 1 in our game, but we can change the rules to allow him to begin with arbitrary capital α.

Write $\mathcal{K}^{\mathcal{P}}$ for his capital process when he begins with zero and follows strategy $\mathcal{P}$:

$$\mathcal{K}^{\mathcal{P}}(\Box) = 0$$

and

$$\mathcal{K}^{\mathcal{P}}(x_1 x_2 \ldots x_n) := \mathcal{K}^{\mathcal{P}}(x_1 x_2 \ldots x_{n-1}) + \mathcal{P}(x_1 x_2 \ldots x_{n-1}) x_n.$$

When he starts with α, his capital process is $\alpha + \mathcal{K}^{\mathcal{P}}$.

The capital processes that begin with zero form a linear space, for

$$\beta \mathcal{K}^{\mathcal{P}} = \mathcal{K}^{\beta \mathcal{P}} \quad \text{and} \quad \mathcal{K}^{\mathcal{P}_1} + \mathcal{K}^{\mathcal{P}_2} = \mathcal{K}^{\mathcal{P}_1 + \mathcal{P}_2}.$$

So the martingales also form a linear space.

CONVEX COMMBINATIONS OF MARTINGALES

Suppose $\mathcal{P}_1$ and $\mathcal{P}_2$ are strategies, and suppose $\alpha_1 + \alpha_2 = 1$. Then

$$\alpha_1(1+\mathcal{K}^{\mathcal{P}_1}) + \alpha_2(1+\mathcal{K}^{\mathcal{P}_2}) = 1 + \mathcal{K}^{\alpha_1\mathcal{P}_1+\alpha_2\mathcal{P}_2}.$$

—LHS is the convex combination of two martingales that begin with capital 1.
—RHS is the martingale produced by the corresponding convex combination of strategies, also beginning with capital 1.

Conclusion:
In the game where we begin with capital 1, we can obtain a convex combination of martingales $1+\mathcal{K}^{\mathcal{P}_1}$ and $1+\mathcal{K}^{\mathcal{P}_2}$ by splitting our capital into two accounts, one with initial capital α_1 and one with initial capital α_2. Apply $\alpha_1\mathcal{P}_1$ (a scaled-down version of $\mathcal{P}_1$) to the first account, and apply $\alpha_2\mathcal{P}_2$ (a scaled-down version of $\mathcal{P}_2$) to the second account.

INFINITE CONVEX COMMBINATIONS OF MARTINGALES

Suppose $\mathcal{P}_1, \mathcal{P}_2, \ldots$ are strategies and $\alpha_1, \alpha_2, \ldots$ are nonnegative real numbers adding to one.

If the sum $\sum_{k=1}^{\infty} \alpha_k \mathcal{P}_k$ converges, then the sum $\sum_{k=1}^{\infty} \alpha_k \mathcal{K}^{\mathcal{P}_k}$ will also converge, by induction on

$$\mathcal{K}^{\mathcal{P}}(x_1 x_2 \ldots x_n) := \mathcal{K}^{\mathcal{P}}(x_1 x_2 \ldots x_{n-1}) \\ +\mathcal{P}(x_1 x_2 \ldots x_{n-1}) x_n.$$

In this case, $\sum_{k=1}^{\infty} \alpha_k \mathcal{K}^{\mathcal{P}_k}$ is the martingale the strategy $\sum_{k=1}^{\infty} \alpha_k \mathcal{P}_k$ produces beginning with capital 1.

Caution:
You can usually get an infinite convex combination of martingales, but you have to check on the convergence of the infinite convex combination of strategies.

FORCING

We say that a strategy $\mathcal{P}$ for Skeptic *forces* an event E if

$$\mathcal{K}^{\mathcal{P}}(t) \geq -1$$

for every t in $\Omega^{\diamond}$ and

$$\lim_{n\to\infty} \mathcal{K}^{\mathcal{P}}(\xi^n) = \infty$$

for every path ξ not in E.

This means $\mathcal{P}$ is a winning strategy in the game where Skeptic starts with capital 1 and has E as his goal instead of $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} x_i = 0$.

Skeptic *can force* E if he has a strategy that forces E. This means there is a nonnegative martingale starting at 1 that becomes infinite on every path not in E.

Such a nonnegative martingale *witnesses* E.

WEAK FORCING

- $\mathcal{P}$ *forces* E if $\mathcal{K}^{\mathcal{P}}(t) \geq -1$ for every t in $\Omega^{\diamond}$ and
$$\lim_{n\to\infty} \mathcal{K}^{\mathcal{P}}(\xi^n) = \infty$$
for every path ξ not in E.

- $\mathcal{P}$ *weakly forces* E if $\mathcal{K}^{\mathcal{P}}(t) \geq -1$ for every t in $\Omega^{\diamond}$ and
$$\sup_n \mathcal{K}^{\mathcal{P}}(\xi^n) = \infty$$
for every path ξ not in E.

We say that Skeptic *can weakly force* E if he has a strategy that weakly forces E. This means that there is a nonnegative martingale starting at 1 that is unbounded on every path not in E.

EQUIVALENCE OF FORCING AND WEAK FORCING

The following lemma shows that forcing and weak forcing are practically equivalent.

Lemma 1 *If Skeptic can weakly force E, then he can force E.*

Proof Suppose $\mathcal{P}$ is a strategy that weakly forces E. For any $C > 0$, define a new strategy $\mathcal{P}^{(C)}$ by

$$\mathcal{P}^{(C)}(s) := \begin{cases} \mathcal{P}(s) \text{ if } \mathcal{K}^{\mathcal{P}}(t) < C \text{ for all } t \sqsubseteq s \\ 0 \text{ otherwise.} \end{cases}$$

This strategy mimics $\mathcal{P}$ except that it quits betting as soon as Skeptic's capital reaches C. Define a strategy $\mathcal{Q}$ by

$$\mathcal{Q} := \sum_{k=1}^{\infty} 2^{-k} \mathcal{P}^{(2^k)}.$$

Then $\lim_{n\to\infty} \mathcal{K}^{\mathcal{Q}}(\xi^n) = \infty$ for every ξ for which $\sup_n \mathcal{K}^{\mathcal{P}}(\xi^n) = \infty$. Since $\mathcal{K}^{\mathcal{P}} \geq -1$, $\mathcal{K}^{\mathcal{Q}} \geq -1$. Since $\sup_n \mathcal{K}^{\mathcal{P}}(\xi^n) = \infty$ for every ξ not in E, $\lim_{n\to\infty} \mathcal{K}^{\mathcal{Q}}(\xi^n) = \infty$ for every ξ not in E. So $\mathcal{Q}$ forces E. ∎

COMBINING STRATEGIES THAT FORCE EVENTS

Lemma 2 *If Skeptic can weakly force each of a sequence $E_1, E_2, \ldots$ of events, then he can weakly force $\bigcap_{k=1}^{\infty} E_k$.*

Proof Suppose $\mathcal{P}_k$ weakly forces E_k. Then

$$|\mathcal{P}_k(x_1 \ldots x_n)| \leq 1 + \mathcal{K}^{\mathcal{P}_k}(x_1 \ldots x_n) \leq 2^n.$$

Skeptic bets $\mathcal{P}_k(x_1 \ldots x_n)$ on round $n+1$. When he makes this bet, he has capital $1 + \mathcal{K}^{\mathcal{P}_k}(x_1 \ldots x_n)$. The first inequality holds because he cannot bet more than he has (he must avoid risking bankruptcy). The second inequality holds because he cannot, consequently, do more than double his money on each round.

Because for each $x_1 \ldots x_n$ there is a constant C (namely 2^n) such that $\mathcal{P}_k(x_1 \ldots x_n) \leq C$ for all k, a strategy $\mathcal{Q}$ can be defined by

$$\mathcal{Q} := \sum_{k=1}^{\infty} 2^{-k} \mathcal{P}_k.$$

Since $\mathcal{P}_k$ weakly forces E_k, $\mathcal{Q}$ also weakly forces E_k. So $\mathcal{Q}$ weakly forces $\bigcap_{k=1}^{\infty} E_k$. ∎

BOUNDING REALITY'S AVERAGE MOVE FROM ABOVE

Lemma 3 *Suppose* $\epsilon > 0$. *Then Skeptic can weakly force*

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i \leq \epsilon.$$

Proof We may suppose that $\epsilon < \frac{1}{2}$. Let $\mathcal{P}$ be the strategy that always buys $\epsilon\alpha$ of Reality's move x, where α is the current capital. Because x is never less than -1, this strategy loses at most the fraction ϵ of the current capital, and hence the capital process is nonnegative. It is given by $1 + \mathcal{K}^{\mathcal{P}}(\Box) = 1$ and

$$1 + \mathcal{K}^{\mathcal{P}}(x_1 \dots x_n) = (1 + \mathcal{K}^{\mathcal{P}}(x_1 \dots x_{n-1}))(1 + \epsilon x_n)$$

$$= \prod_{i=1}^{n} (1 + \epsilon x_i) .$$

Let $\xi = x_1 x_2 \dots$ be a path such that

$$\sup_n \mathcal{K}^{\mathcal{P}}(x_1 \dots x_n) < \infty.$$

Then there exists a constant $C_\xi > 0$ such that

$$\prod_{i=1}^{n} (1 + \epsilon x_i) \leq C_\xi$$

for all n. This implies that

$$\sum_{i=1}^{n} \ln(1+\epsilon x_i) \leq D_\xi$$

for all n for some D_ξ. Since $\ln(1+t) \geq t - t^2$ whenever $t \geq -\frac{1}{2}$, ξ also satisfies

$$\epsilon \sum_{i=1}^{n} x_i - \epsilon^2 \sum_{i=1}^{n} x_i^2 \leq D_\xi,$$

$$\epsilon \sum_{i=1}^{n} x_i - \epsilon^2 n \leq D_\xi,$$

$$\epsilon \sum_{i=1}^{n} x_i \leq D_\xi + \epsilon^2 n,$$

or

$$\frac{1}{n} \sum_{i=1}^{n} x_i \leq \frac{D_\xi}{\epsilon n} + \epsilon$$

for all n and hence satisfies

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i \leq \epsilon.$$

Thus $\mathcal{P}$ weakly forces this event. ∎

BOUNDING REALITY'S AVERAGE MOVE FROM BELOW

The same argument, with $-\epsilon$ in place of ϵ, establishes the following complementary lemma.

Lemma 4 *Suppose $\epsilon > 0$. Then Skeptic can weakly force*

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i \geq -\epsilon.$$

PROOF OF THE GAME-THEORETIC STRONG LAW

We must show that Skeptic can force

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i = 0. \tag{1}$$

According to Lemma 1, it suffices to show he can weakly force (1).

Consider the events

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i \leq \epsilon$$

and

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i \geq -\epsilon$$

for $\epsilon = 2^{-k}$, where k ranges over all natural numbers. By Lemmas 3 and 4, Skeptic can weakly force each of these events. By Lemma 2, he can therefore weakly force their intersection, which is (1).

THE CONSTRUCTIVE NATURE OF THE PROOF

- The core idea is to fix an integer k and to pursue a strategy that always sets one's move M_n equal to $1/k$ of one's current capital. This strategy is easy to implement. It is computable in every reasonable sense. It is measurable.

- A secondary idea is to combine a countable number of such strategies. The combination is also computable and measurable.

- Another secondary idea is convert a strategy that weakly forces something into a strategy that forces it by applying a countable number of stopping rules and then recombining. This is also computable and measurable.

Because Skeptic's strategy is computable and measurable, his capital process (martingale) is also computable and measurable. This is clear from the formula that derives the martingale from the strategy inductively:

$$\begin{aligned}\mathcal{K}^{\mathcal{P}}(x_1x_2\ldots x_n) &:= \mathcal{K}^{\mathcal{P}}(x_1x_2\ldots x_{n-1}) \\ &+\mathcal{P}(x_1x_2\ldots x_{n-1})x_n.\end{aligned}$$

GAME-THEORETIC STRONG LAW $\Rightarrow$ MEASURE-THEORETIC STRONG LAW

Corollary 1 *Suppose* $x_1, x_2, \ldots$ *is an adapted sequence of random variables in a filtered probability space* $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n=0}^{\infty}, \mathbb{P})$. *Suppose* $-1 \leq x_n \leq 1$ *and* $\mathbb{E}[x_n|\mathcal{F}_{n-1}] = 0$ *for all* n. *Then* $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i = 0$ *almost surely.*

Proof Suppose Reality plays the random variables $x_1, x_2, \ldots$ as his moves in our game. Suppose Skeptic follows his measurable winning strategy $\mathcal{P}$. Then Skeptic's capital process, a game-theoretic martingale, can also be thought of as a sequence of random variables: $\mathcal{K}_0 = 1$ and

$$\mathcal{K}_n(x_1 \ldots x_n) = \mathcal{K}_{n-1}(x_1 \ldots x_{n-1}) + \mathcal{P}(x_1 \ldots x_{n-1})x_n.$$

Because $\mathbb{E}[x_n|\mathcal{F}_{n-1}] = 0$, $\mathcal{K}_0, \mathcal{K}_1, \ldots$ is also a measure-theoretic martingale. It is nonnegative and diverges to infinity if $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i = 0$ fails. By Doob's martingale convergence theorem, a nonnegative measure-theoretic martingale diverges to infinity with probability zero. So $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i = 0$ happens almost surely. ∎

MEASURE-THEORETIC STRONG LAW $\not\Rightarrow$ GAME-THEORETIC STRONG LAW

The measure-theoretic strong law is a corollary of the game-theoretic strong law because a meaurable game-theoretic martingale is necessarily a measure-theoretic martingale when Reality's x_n are random with conditional mean zero.

We cannot go the other way. When we begin with the measure-theoretic strong law, we know only that there is a nonnegative measure-theoretic martingale that diverges to ∞ when $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i = 0$ fails. Nothing tells us that it is a *game-theoretic* martingale. (Nothing tells us that its nth increment is conditionally a multiple of x_n.)

In order to derive the game-theoretic strong law, we need more than the *statement* of the measure-theoretic strong law. We must look inside the *proof*, as we have done in this lecture.