The Game-Theoretic Capital Asset Pricing Model

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February 11, 2002

- The classical capital asset pricing model (CAPM)
- The game-theoretic CAPM: Informal statement (game not spelled out)
- A game for asset pricing
- Upper probability
- The game-theoretic CAPM: Formal statement
- The game-theoretic CAPM: Formal proof

TERMINOLOGY AND NOTATION

Consider a security whose value is v_1 at time 0 and v_1 at time 1.

Write s for the *simple return*:

$$s := \frac{v_1 - v_0}{v_0}.$$

ASIDE

We can also write

$$v_1 = v_0(1 + s) \qquad \text{and} \qquad 1 + s = \frac{v_1}{v_0}.$$

Later we will consider the *logarithmic return*,

$$\ln(1 + s) = \ln\left(\frac{v_1}{v_0}\right).$$

THE CLASSICAL CAPM

Suppose we adopt a stochastic model for a stock, a market index, and a risk-free bond.

- $\tilde{s}$ is the random variable whose realization is the simple return s for the stock.

- $\tilde{m}$ is the random variable whose realization is the simple return m for the market index.

- f is the simple return (interest rate) for the risk-free bond. (This is known in advance and therefore not random.)

The classical capital asset pricing model (CAPM) says that

$$\mathsf{E}(\tilde{s}) = f + (\mathsf{E}(\tilde{m}) - f)\frac{\mathsf{Cov}(\tilde{s}, \tilde{m})}{\mathsf{Var}(\tilde{m})}.$$

CLASSICAL CAPM

$$\mathsf{E}(\tilde{s}) = f + (\mathsf{E}(\tilde{m}) - f)\beta_s,$$

where β_s is the slope of the theoretical linear regression of $\tilde{s}$ on $\tilde{m}$.

This formula, which dates from the 1960s, is widely used in theoretical and applied finance. It shows up repeatedly in textbooks for every topic in finance. Some of its lessons are:

- The risk of a stock is measured by its covariance with the market, not by its variance. The part of its variance that is orthogonal to the market can be diversified away.

- There is a tradeoff between the risk β_s of a stock and its expected return $\mathsf{E}(\tilde{s})$. To get a higher return, you must take on more risk.

- Any risk/return profile can be obtained by holding a mixture of the risk-free bond and the entire market. (To get risk β, invest β of your money in the market and $1 - \beta$ in the risk-free bond.) There is no need to hold individual stocks.

WHAT DOES THE STATISTICAL MODEL MEAN?

There is a lot of waffling on this question.

- Theoretically, the probabilities are those of investors. The model is a *pricing model*, because theoretically, investors can use the formula to price a stock or investment. They estimate decide how risky a stock or investment is (β_s) and then calculate the price $\mathsf{E}(\tilde{s})$ it should have.

- If the market is efficient, then the prices investors pay should be reflected in actual returns. So the probabilities should also work as frequencies/objective probabilities. This provides a basis for statistical tests.

- When the model does poorly statistically, we can still claim that it models what investors did. They just didn't get it right.

Theorists in finance now regard CAPM as a flawed special case of a more general theory. But the general theory preserves CAPM's equivocation about the meaning of the stochastical model. The probabilities are supposed to be both (1) subjective probabilities of investors and (2) "physical probabilities" that govern the market. Whenever the conversation focuses on considerations that make one of the two interpretations implausible, the other interpretation is used.

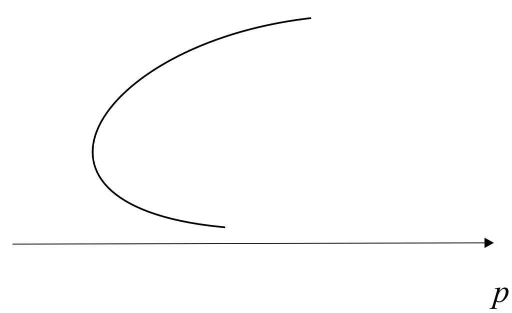
$$\boxed{\mathsf{E}(\tilde{s}) = f + (\mathsf{E}(\tilde{m}) - f)\beta_s}$$

HOW IS THE CLASSICAL CAPM DERIVED?

Reference: Copeland and Weston (1988), p. 195.

Consider the various portfolios that can be held in a securities market. For each portfolio, consider the mean $\mathsf{E}(\tilde{p})$ and volatility (standard deviation) σ_p of the simple return p.

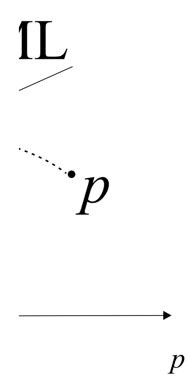
We may suppose that the efficient frontier (smallest volatility for each level of expected return) looks like this:



If investors' utilities can be described in terms of mean and variance (strong assumption) and either returns have a normal distributions or else everyone has the same variance-return tradeoff (very strong and unreasonable assumptions), then the market portfolio will be on the efficient frontier.

By mixing the market portfolio m with the risk-free bond f, you can achieve any point on the capital market line (CML):

Consider a portfolio p inside the opportunity set, and the (volatility-mean) trajectory formed by mixing p with m:



This trajectory must be tangent to the efficient frontier (and hence also to the CML) at m. Otherwise, you could extend the trajectory past the efficient frontier by going a little short in p in order to go longer in m.

The mathematical condition that the trajectory be tangent is $\mathsf{E}(\tilde{p}) = f + (\mathsf{E}(\tilde{m}) - f)\beta_p$.

Theorists in finance agree that the classical CAPM is flawed by its strong assumptions about the beliefs and preferences of investors. So they have developed a more general theory, which allows more general assumptions about beliefs and preferences.

But the more general theory still adopts a stochastic model and preserves the classical CAPM's equivocation about its meaning. The probabilities are supposed to be both (1) subjective probabilities of investors and (2) "physical probabilities" that govern the market. Whenever conversation focuses on considerations that make one of the two interpretations implausible, the other interpretation is used.

The game-theoretic CAPM is more radical. It drops altogether the assumption that there is a stochastic model.

1. We make no assumption about the beliefs and preferences of investors.

2. We do not assume that some "physical probabilities" govern security prices.

TOWARDS THE GAME-THEORETIC CAPM

Instead of considering someone's probabilities for a one-period return, consider actual returns over N periods:

- $s_1, \ldots, s_N$ are the simple returns for a security s.
- $m_1, \ldots, m_N$ are the simple returns for a specified market index m.

These returns take account of the total gain or loss (capital gain or loss plus dividends and redistributions) from holding the stock or index.

We use these symbols for the averages and *uncentered* empirical variance and covariance:

$$\mu_s := \frac{1}{N}\sum_{n=1}^{N} s_n, \qquad \mu_m := \frac{1}{N}\sum_{n=1}^{N} m_n,$$
$$\sigma_m^2 := \frac{1}{N}\sum_{n=1}^{N} m_n^2, \qquad \sigma_{sm} := \frac{1}{N}\sum_{n=1}^{N} s_n m_n.$$

These are empirical (ex post) quantities, not theoretical (ex ante) quantities like $\mathsf{E}(\tilde{s})$ and $\mathsf{Cov}(\tilde{s}, \tilde{m})$.

We will also now use the symbol β_s for an empirical quantity: $\beta_s := \sigma_{sm}/\sigma_m^2$.

LOGARITHMIC RETURNS

The empirical growth of an investment in s is best measured not by averaging the simple returns s_n but by averaging the logarithmic returns $\ln(1 + s_n)$.

If you invest \$1 in s and reinvest all dividends, then your terminal wealth at the end of N periods is

$$W_s := \prod_{n=1}^{N} (1 + s_n).$$

So

$$\frac{1}{N} \ln W_s = \frac{1}{N} \ln \prod_{n=1}^{N} (1 + s_n) = \frac{1}{N} \sum_{n=1}^{N} \ln(1 + s_n).$$

The average of the logarithmic returns is simply $1/N$ times the log of the terminal wealth.

How is the terminal wealth related to the average of the simple returns?

The Taylor expansion $\ln(1 + x) \approx x - \frac{1}{2}x^2$ yields

$$\boxed{\frac{1}{N} \ln W_s \approx \frac{1}{N} \sum_{n=1}^{N} \left(s_n - \frac{1}{2} s_n^2 \right) = \mu_s - \frac{1}{2} \sigma_s^2.}$$

We call $\frac{1}{N} \ln W \approx \mu - \frac{1}{2}\sigma^2$ the *fundamental approximation of asset pricing*.

FUNDAMENTAL APPROXIMATION

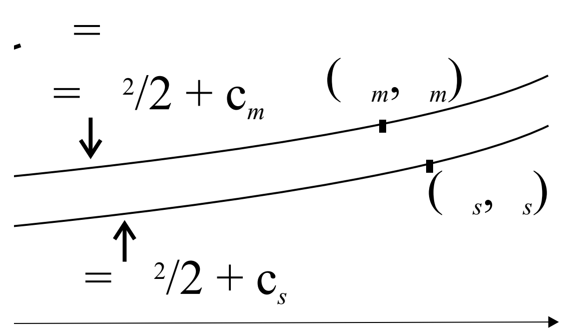
$$\frac{1}{N}\ln W \approx \mu - \frac{1}{2}\sigma^2$$

This formula tells us that volatility reduces the attractiveness of a given average simple return regardless of

- whether volatility is an indication of risk,
- whether an investor is risk averse,
- whether returns are driven by a stochastic mechanism or a game.

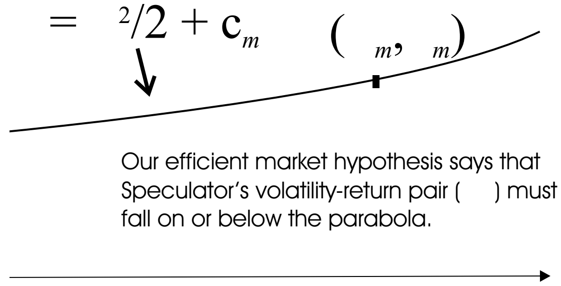
We prefer smaller σ for given μ simply because we prefer greater terminal wealth!!

The preference for greater terminal wealth creates indifference curves in the σ, μ plane:

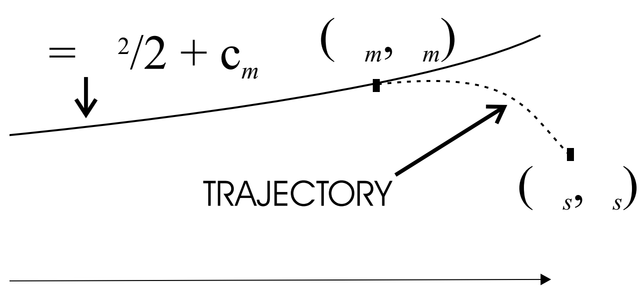


THE GAME-THEORETIC CAPM

The game-theoretic CAPM is based on the hypothesis that a speculator cannot beat m. In other words, he cannot get above the indifference parabola $\mu = \sigma^2 + c_m$:



Consider a portfolio s and the trajectory formed by portfolios that mix s and m:



This trajectory must be tangent to the indifference parabola at (σ_m, μ_m). Otherwise, a speculator can get above the indifference parabola by going short in s to go longer in m.

The mathematical condition that the trajectory be tangent is $\mu_s \approx \mu_m - \sigma_m^2 + \sigma_{sm}$, or $\mu_s \approx (\mu_m - \sigma_m^2) + \sigma_m^2 \beta_s$.

CLASSICAL CAPM

$$\mathsf{E}(\tilde{s}) = f + (\mathsf{E}(\tilde{m}) - f)\beta_s$$

GAME-THEORETIC CAPM

$$\mu_s \approx (\mu_m - \sigma_m^2) + \sigma_m^2 \beta_s$$

If we write μ_f for $\mu_m - \sigma_m^2$, then the game-theoretic CAPM can be written in the form

$$\mu_s \approx \mu_f + (\mu_m - \mu_f)\beta_s.$$

Some ways the game-theoretic CAPM differs from the classical CAPM:

1. It replaces theoretical expected values, variances, and covariances with empirical quantities. (The game-theoretic model has no probability measure and therefore no such theoretical quantities.)

2. It replaces assumptions about beliefs and preferences (and/or assumptions about stochasticity) with a very concrete efficient market hypothesis.

3. It replaces an exact equation between theoretical quantities with an approximate equation between empirical quantities.

4. It replaces the risk-free rate of return with $\mu_m - \sigma_m^2$.

CLASSICAL CAPM

$$\mathsf{E}(\tilde{s}) = f + (\mathsf{E}(\tilde{m}) - f)\beta_s$$

GAME-THEORETIC CAPM

$$\mu_s \approx \mu_f + (\mu_m - \mu_f)\beta_s,$$

where $\mu_f := \mu_m - \sigma_m^2$.

More fundamentally:

- We might imagine using the classical equation to determine a price for s. The game-theoretic formula, in contrast, is not a rule for pricing capital assets in advance. It is an *ex post* rather than an *ex ante* model.

- The classical equation cannot really be tested, because it is about theoretical quantities. The game-theoretic formula, as we shall see, can be tested, because a precise error bound can be derived from the fundamental approximation and the efficient market hypothesis.

Basic Capital Asset Pricing Game

Players: Investor, Market, Speculator

Parameters:

Natural number K (# of non-index securities)
Natural number N (# of rounds or trading periods)
Real number α satisfying $0 < \alpha \leq 1$ (significance level)
$A \subseteq \Omega$ (auxiliary goal)

Protocol:

$\mathcal{G}_0 := 1$.
$\mathcal{H}_0 := 1$.
$\mathcal{M}_0 := 1$.
FOR $n = 1, 2, \ldots, N$:
  Investor selects $g_n \in \mathbb{R}^{K+1}$ such that $\sum_{k=0}^{K} g_n^k = 1$.
  Speculator selects $h_n \in \mathbb{R}^{K+1}$ such that $\sum_{k=0}^{K} h_n^k = 1$.
  Market selects $x_n \in (-1, \infty)^{K+1}$.
  $\mathcal{G}_n := \mathcal{G}_{n-1} \sum_{k=0}^{K} g_n^k (1 + x_n^k)$.
  $\mathcal{H}_n := \mathcal{H}_{n-1} \sum_{k=0}^{K} h_n^k (1 + x_n^k)$.
  $\mathcal{M}_n := \mathcal{M}_{n-1} (1 + x_n^0)$.

Winner: Speculator wins if $\mathcal{H}_n \geq 0$ for $n = 1, \ldots, N$ and either (1) $\mathcal{H}_N \geq \frac{1}{\alpha} \mathcal{M}_N$ or (2) $(g_1, x_1, \ldots, g_N, x_N) \in A$. Otherwise Investor and Market win.

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If Speculator has a winning strategy, then we say that the efficient market hypothesis (EMH) for m predicts A at level α.

We call the number $\underline{\mathbb{P}}\,A$ defined by

$$\underline{\mathbb{P}}\,A := 1 - \alpha_A,$$

where

$\alpha_A := \inf \{\alpha \mid 0 < \alpha \leq 1 \text{ and EMH predicts } A \text{ at level } \alpha\}$,

the *lower probability* for A.

Roughly speaking (neglecting the fact that the infimum might not be attained), the lower probability of A is the degree of belief corresponding to the smallest α such that A is predicted at level α. Because Speculator has a winning strategy for the goal A when $\alpha = 1$ (buy and hold security 0), we always have $0 \leq \alpha_A \leq 1$ and hence $0 \leq \underline{\mathbb{P}}\,A \leq 1$.

Writing A^{c} for A's complement ($\Omega \setminus A$), we set

$$\overline{\mathbb{P}}\,A := 1 - \underline{\mathbb{P}}\,A^{\mathsf{c}},$$

and we call $\overline{\mathbb{P}}\,A$ the *upper probability* of A. This quantity measures how plausible A is—the degree to which there is no particular reason for believing its complement A^{c}. It can be shown, using the fact that Speculator cannot make money for sure in the game, that $\underline{\mathbb{P}}\,A \leq \overline{\mathbb{P}}\,A$.

The *long-short CAPG* has two extra parameters: a positive constant C (perhaps very large), and a positive constant δ (perhaps very small). It is obtained by replacing the condition $g_n \in \mathbb{R}^{K+1}$ in the protocol for the basic CAPG by the condition $g_n \in [0,\infty)^{K+1}$ and replacing the condition $x_n \in (-1,\infty)^{K+1}$ by the conditions $x_n \in (-1,C]^{K+1}$ and $m_n \geq -1+\delta$. (Remember that $m_n = x_n^0$.) In other words, Investor is not allowed to sell short, and Market is constrained so that an individual security cannot increase too much in value on a single round and the market index m cannot lose too much of its value on a single round. These constraints on Investor and Market make it possible for Speculator to go short in Investor's moves, at least a bit, without risking bankruptcy.

Proposition 1 *For any $\epsilon \in \left(0, \frac{\delta}{1+C}\right)$ and $\alpha \in (0,1]$, the efficient market hypothesis for m predicts that*

$$\left|\mu_s - \mu_m + \sigma_m^2 - \sigma_{sm}\right| < \frac{E}{\epsilon} + \frac{\ln\frac{2}{\alpha}}{N\epsilon} + \frac{\epsilon}{2}\sigma_{s-m}^2$$

at level α in the long-short CAPG with parameters C and δ, where

$$E := \max_{j\in\{-1,1\}} \frac{1}{N}\sum_{n=1}^{N}\left(\Gamma(m_n) - \gamma((1-j\epsilon)m_n + j\epsilon s_n)\right)$$

and the functions Γ and γ are defined by

$$\Gamma(x) := \frac{1}{3}x^3, \qquad \gamma(x) := \frac{1}{3}\left(\frac{x}{1+x}\right)^3.$$