Bringing betting games back to the center of probability and statistics

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Testing and estimation using betting, e-values and martingales
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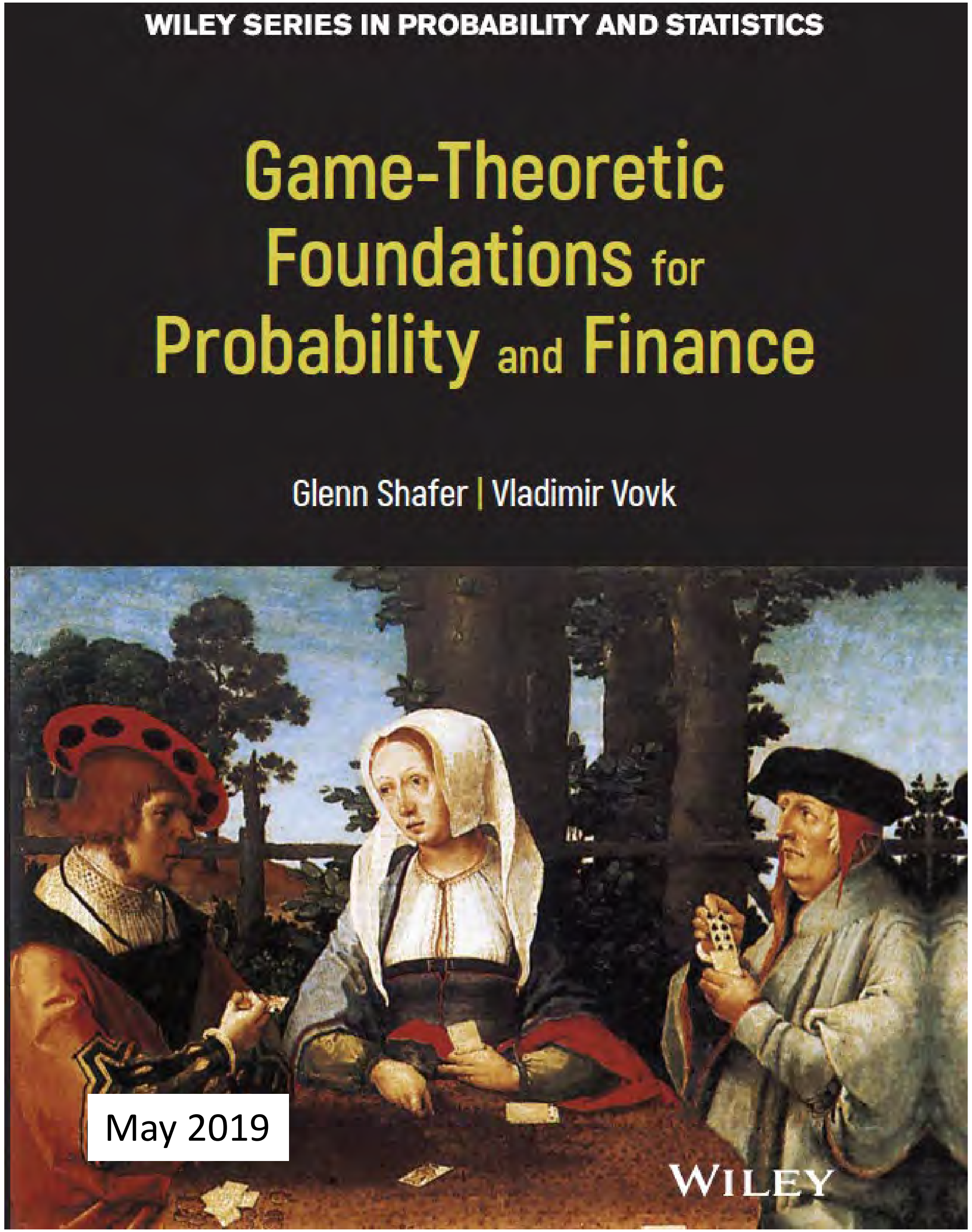


Base mathematical probability on testing by betting.

Papers at www.probabilityandfinance.com:

- 47 (efficient markets)
- 54 (testing by betting)
- 55 (history of testing)
- 56 (statistics)
- 57 (random risk)

1. Pascal and Huygens began with betting games.
2. Bernoulli suppressed the game to make probabilities seem objective.
3. De Finetti brought back the player who offers odds.
4. Shafer and Vovk use 3 players:

    Forecaster gives odds.

    Skeptic selects bets.

    Reality decides the outcome.

5. The unity and diversity of probability

1. Pascal and Huygens began with betting games.

In 1654, Pascal announced he would write a book:

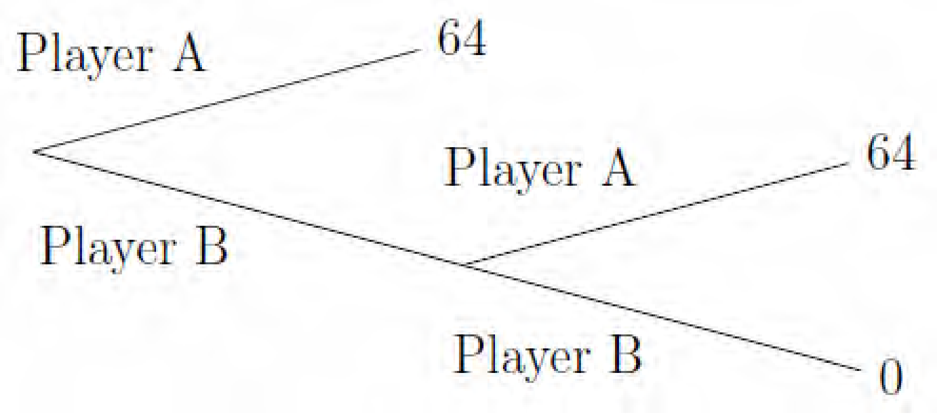
The uncertainty of fate is so well overcome by the rigor of calculation that each player is assigned exactly the amount they deserve.

This can only be found by reasoning, not by experience.

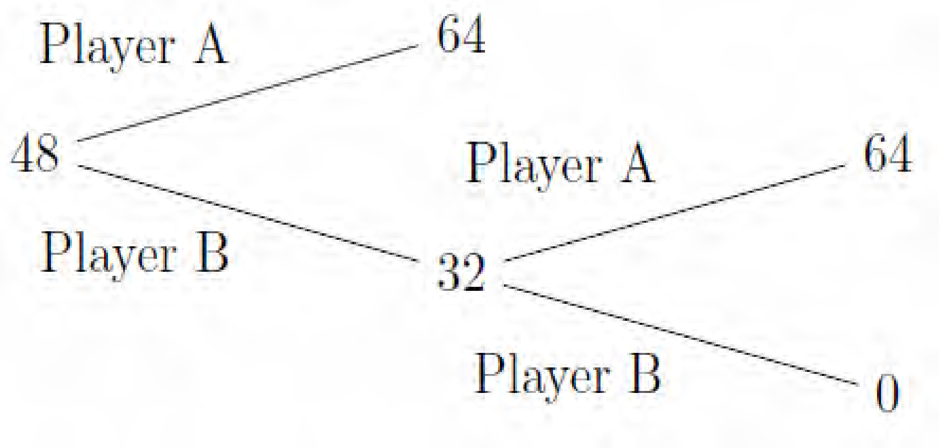
Book's title will be *Geometry of Chance*.

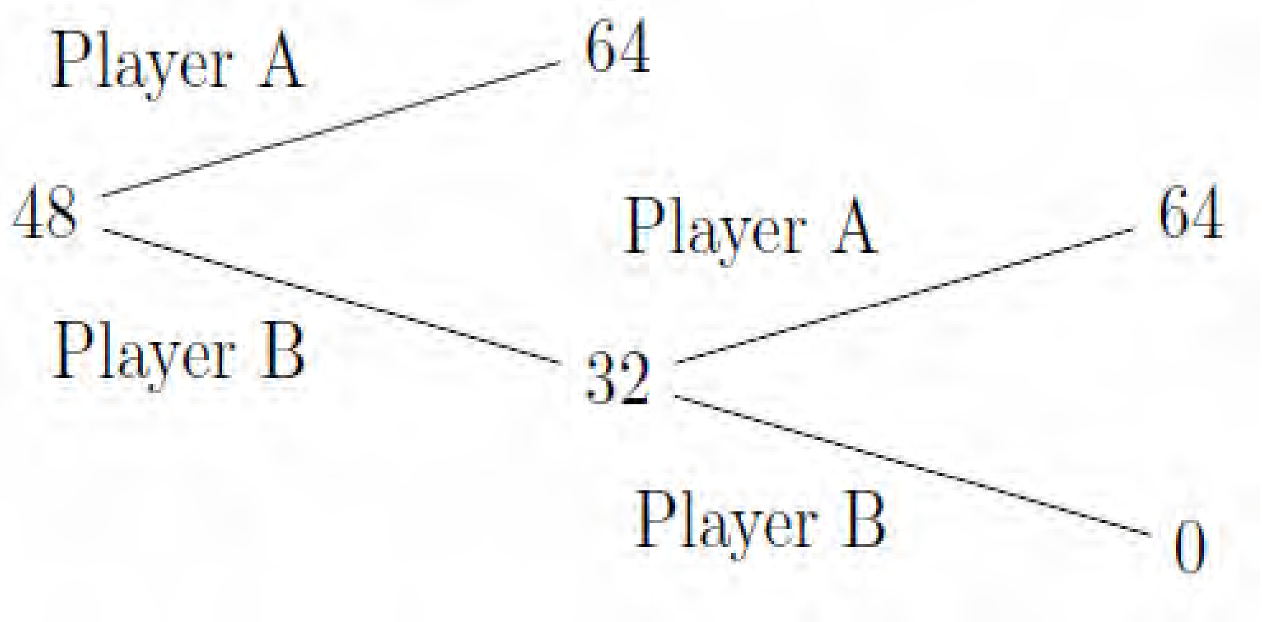
Blaise Pascal
1623 - 1662

Pascal’s division problem



Pascal’s solution





Pascal's two principles:

1. Take any amount you are sure to get.

2. *If the game is one of pure chance*, and the chances for winning a certain amount are equal, then divide the amount equally.

Teachers of commercial arithmetic used Pascal's argument in 1400s.

- They did <u>not</u> assume pure chance.

- Only assumed play on equal terms.

Proposition I in Huygens's book.

If I have the same chance to get a or b, it is worth as much to me as $(a + b)/2$.

Consider this fair game:

- We both stake x.
- The winner will give a to the loser.

The analysis:

- If I win, I get $2x - a$.
- If this is equal to b, then $x = (a + b)/2$.

The synthesis:

- Having $(a + b)/2$, I can play with an opponent who stakes the same amount, on the understanding that the winner gives the loser a.
- This gives me equal chances of getting a or b.

Christiaan Huygens
1629 - 1695

2. Bernoulli suppressed the game to make probabilities seem objective.

Jacob Bernoulli (1654-1705),
painted by his younger brother Nicolaus

JACOBI BERNOULLI,
Profess. Basil. & utriusque Societ. Reg. Scientiar.
Gall. & Pruss. Sodal.
MATHEMATICI CELEBERRIMI,
ARS CONJECTANDI,
OPUS POSTHUMUM.
Accedit
TRACTATUS
DE SERIEBUS INFINITIS,
Et EPISTOLA Gallicè scripta
DE LUDO PILÆ
RETICULARIS.

FESTINA LENTE

BASILEÆ,
Impensis THURNISIORUM, Fratrum.
cIↃ IↃCC XIII.

Published in 1713
by his nephew Nicolaus

Bernoulli did not like Huygens's game-theoretic proof.

He wrote,

> I will use more popular and easily understood reasoning.
>
> I assume only this:
>
> > *You may expect just as much as you will get for sure.*

Then we get the circular "equal chances" talk.

3. De Finetti brought back the player who offers odds.

Bruno de Finetti
1906-1985

“You have subjective probability p for A”

What does it mean?

Bertrand 1889	Borel 1924	De Finetti 1931
You would trade the consequences of A for the same consequences of an event with objective probability p.	Choose one: • I give you 100 francs if A happens. • I give you 100 francs if an event with objective probability 1/3 happens.	Choose p such that you are willing to offer odds p:(1-p) on A to all comers.
Caveat: If this assimilation is impossible, don’t do it.	Caveats: Imprecise in exceptional cases. More importantly, in a game (e.g., poker, election) your bet might affect the outcome.	Elaboration: You should order events from less to more probable. Axioms on the ordering justify the rules of probability. The betting scheme is an imperfect *operationalization*.

De Finetti cited Bertrand but did not mention Borel.

4. Shafer and Vovk use 3 players:

Forecaster gives odds.

Skeptic selects bets.

Reality decides the outcome.

Forecaster gives odds.

Skeptic selects bets.

Reality decides outcome.

Testing a forecaster

Forecaster announces a probability distribution P on $\mathcal{Y}$.
Skeptic announces $S : \mathcal{Y} \to [0, \infty)$ such that $\mathbf{E}_P(S) = 1$.
Reality announces $y \in \mathcal{Y}$.
$\mathcal{K} := S(y)$.

Skeptic multiplied the money he risked by $\mathscr{K}$.

Forecaster is discredited if $\mathscr{K}$ is very large.

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Markov's inequality: Once S is chosen, P's probability for $\mathcal{K}$ being larger than K is no more than $1/K$.

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Markov's inequality: Once S is chosen, P's probability for $\mathcal{K}$ being larger than C is no more than $1/C$.

- Skeptic can play so that the capital he risks is multiplied by $1/P(E)$ if E happens, but this is the best he can do.
- So P's probabilities tell what Skeptic can do.
- **What Skeptic can do defines the probabilities.**

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Useful reformulation

Testing a forecaster

Skeptic announces $\mathcal{K}_0$.

Forecaster announces a probability distribution P on $\mathcal{Y}$.
Skeptic announces $S : \mathcal{Y} \to \mathbb{R}$ such that $\mathbf{E}_P(S) = \mathcal{K}_0$.
Reality announces $y \in \mathcal{Y}$.
$\mathcal{K}_1 := S(y)$.

Testing a forecaster

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- $P(E)$ is the least initial capital $\mathcal{K}_0$ Skeptic needs to risk to guarantee 1 when E happens.
- $E_P(X)$ is the least initial capital $\mathcal{K}_0$ (positive or negative or zero) Skeptic needs to guarantee X no matter what Reality does.

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Testing a forecaster

Skeptic announces $\mathcal{K}_0$.

FOR $n = 1, 2, \ldots, N$:

Forecaster announces a probability distribution P_n on $\mathcal{Y}$.

Skeptic announces $S_n : \mathcal{Y} \to \mathbb{R}$ such that $\mathbf{E}_{P_n}(S_n) = \mathcal{K}_{n-1}$.

Reality announces $y_n \in \mathcal{Y}$.

$\mathcal{K}_n := S_n(y_n)$.

Generalization

What Skeptic can do defines upper and lower probabilities for what Forecaster and Reality will do.

Testing a forecaster

Forecaster announces a probability distribution P on $\mathcal{Y}$.
Skeptic announces $S : \mathcal{Y} \to [0, \infty)$ such that $\mathbf{E}_P(S) = 1$.
Reality announces $y \in \mathcal{Y}$.
$\mathcal{K} := S(y)$.

Psychological shift:
P belongs to Forecaster,
not to Reality.

Testing a forecaster

Skeptic announces $\mathcal{K}_0$.

FOR $n = 1, 2, \ldots, N$:

Forecaster announces a probability distribution P_n on $\mathcal{Y}$.

Skeptic announces $S_n : \mathcal{Y} \to \mathbb{R}$ such that $\mathbf{E}_{P_n}(S_n) = \mathcal{K}_{n-1}$.

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In this sequential generalization, Forecaster does not announce a probability distribution on $\mathcal{Y}^N$.

A probability distribution on $\mathcal{Y}^N$ is a strategy for Forecaster.

Again, the probability distribution belongs to Forecaster, not to Reality.

Probability belongs to Forecaster, not Reality.

Because Forecaster can use it as a strategy.

Wait!

Skeptic can also use a probability distribution as a strategy (Kelly betting).

- A bet S is a function of Y satisfying $S \geq 0$ and $\sum_y S(y)P(y) = 1$.
- So SP is also a probability distribution. Call it the alternative Q.
- But $Q(y) = S(y)P(y)$ implies $S(y) = Q(y)/P(y)$.
- A bet against P defines an alternative Q and the betting score $S(y)$ is the likelihood ratio $Q(y)/P(y)$.

See my JRSS A paper, "Testing by betting".

5. The unity and diversity of probability

What do different interpretations of probability have in common?

- Conventional answer, going back to Ernest Nagel in 1939: Kolmogorov’s axioms.
- My answer (since the 1980s): a game.

There are many ways to use the probability game.

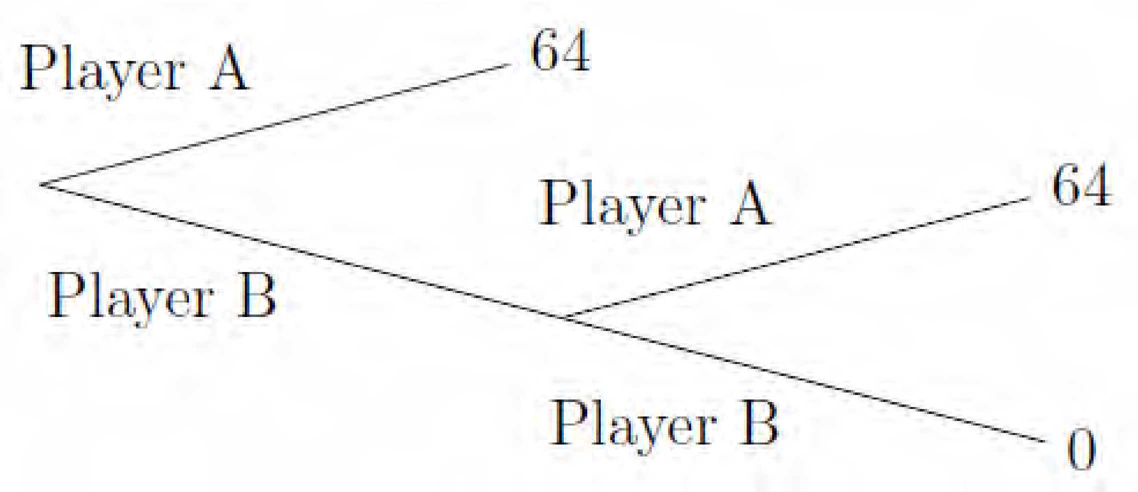
Some ways of using the probability game

1. Prove theorems of mathematical probability.
2. Express beliefs.
3. Express a theory's predictions.
4. Test a theory's predictions.
5. Test a forecaster's predictions.
6. Test market efficiency.
7. Make good probability predictions.
8. Study the properties of prices produced by speculation.
9. Price financial options.
10. Study number theory.
11. Study cryptography.

Extra slide

Blaise Pascal
1623-1662

Pierre Fermat
1607-1665



Fermat's solution: There are four equal chances:

Player A wins the first round, Player A wins the second round;
Player A wins the first round, Player B wins the second round;
Player B wins the first round, Player A wins the second round;
Player B wins the first round, Player B wins the second round.

Player A wins in 3 of the 4 chances and so should get ¾ of the stakes.

Pascal's response: My solution is better because **it carries its demonstration in itself**.
(Your solution requires experience of frequencies.)

Pascal's letters to Fermat

- Written in 1654
- Published in 1679

Pascal's *Triangle arithmétique*

- Written at the end of 1654
- Published in 1665
- Rare

Blaise Pascal
1623 - 1662

Huygens's *De Ratiociniis in ludo aleae*.

- Inspired by 1655 visit to Paris
- Drafted 1656
- Published 1657
- Widely distributed and translated

Christiaan Huygens
1629 - 1695

De Finetti's reservations about betting

1931

Some people dislike betting; others are attracted.

The betting scheme is only an imperfect "operationalization".

1960s

People taking bets often have private information.

"Another shortcoming of the definition—or of the device for making it operational—is the possibility that people accepting bets against our individual have better information than he has (or know the outcome of the event considered). This would bring us to game-theoretic situations."

Note added to translation from the French (in Kyburg and Smokler 1964) of 1937 French article

De Finetti defined probability this way in the 1970s:

We might ask an individual, e.g., You, to specify the *certain gain* which is considered *equivalent* to X. This we might call the price (for You) of X (we denote it by $\mathbf{P}(X)$) in the sense that, on your scale of preference, the random gain X is, or is not, preferred to a certain gain x according as x is less than or greater than $\mathbf{P}(X)$.

...the possibility of inserting the degree of preferability of a random gain into the scale of the certain gains is obviously a prerequisite condition of all decision-making criteria.

Theory of Probability, Volume 1, 1974, page 73.

Here X might be negative. De Finetti called $\mathbf{P}(X)$ its *prevision*. The more traditional name in English is *expected value*.