Maurice Fréchet named Cournot’s principle.

What did he name?

Glenn Shafer, Rutgers University

Maurice Fréchet: Les mathématiques, l'abstrait et le concret, http://conferencefrechet2023.fr

Institut Henri Poincaré, Paris

October 10, 2023

These slides are at www.glennshafer.com, talk # 207.

This talk is dedicated to Thierry Martin, whose *Probabilités et critique philosophique selon Cournot* introduced me and many others to the complexity of Cournot’s principle.

Maurice Fréchet named Cournot’s principle.
What did he name?

Jacob Bernoulli inherited from the scholastics the principle that sufficiently high probability provides practical certainty.

Antoine Augustin Cournot’s originality was to see the principle as a bridge between mathematics and the objective world.

The naming began with Aleksandr Chuprov and culminated with Fréchet.

Fréchet advanced two opinions of his own:

- Cournot’s principle supports Fréchet’s own conception of probability as a physical quantity.
- The event that is impossible because of its small probability must be specified in advance.

From moral to physical certainty

Cournot, 1843

. . . The physically impossible event is therefore the one that has infinitely small probability, and only this remark gives substance, an objective and phenomenal value to the theory of mathematical probability.

. . . L'événement physiquement impossible est donc celui dont la probabilité mathématique est infiniment petite; et cette seule remarque donne une consistance, une valeur objective et phénoménale à la théorie de la probabilité mathématique.

Exposition de la théorie des chances et des probabilités, §43

Cournot clarifies "infinitely small", 1875

In practice, moreover, and in the world of realities, what mathematicians call an infinitely small probability is an exceedingly small probability. The tip of this very sharp needle is not a mathematical point like the apex of the cone in question. Viewed through a magnifying glass, it becomes a blunt tip. With whatever care we polish the plane of steel or agate on which we try to balance it, very delicate experiments will show roughness and streaks. It follows that the probability of success in putting the needle in equilibrium is no longer infinitely small. It is only excessively small, as would be the probability of rolling an ace a hundred times with an unloaded die, which is enough for us to judge, with no fear of being refuted by experience, that the equilibrium is physically impossible.

The same remarks apply to the market value of commercial chances. . . .

Matéralisme, Vitalisme, Rationalisme: Études sur l'emploi des données de la science en philosophie, §IV.4

Question

Infinitely small = Very small.   So what's the difference between

- Cournot's physical impossibility/certainty and
- his predecessors' moral impossibility/certainty?

Some of the predecessors: Molina, Descartes, Locke, Jacob Bernoulli, Buffon, d'Alembert, Condorcet, Laplace, Fourier, Poisson.

Answer

Cournot used the concept as a bridge between

- pure mathematics (probability) and
- the physical world.

For the 18th century pre-Kantian predecessors, there was no chasm to bridge.

Caveat

Much earlier, Condorcet already understood moral certainty as a bridge between

- probability mathematics and
- practical conclusions.

Cournot differed from Condorcet as to what was on the other side of the bridge from the mathematics.

Condorcet’s eulogy of Buffon, 1790

- Mr. de Buffon proposed that we assign a precise value to the very large probability that we can consider moral certainty, and beyond this to ignore the small possibility of a contrary event.
- This principle is true when we only want to make ordinary use of a calculation; and in this sense all men have adopted it in practice and all philosophers have followed it in their reasoning.
- But it ceases to be correct if we introduce it into the calculus itself, and especially if we want to use it to establish theories, explain paradoxes, and prove or refute general rules.

Condorcet's eulogy continued

- Besides, this probability, which may be called moral certainty, must be greater or smaller according to the nature of the objects considered and the principles that should guide our conduct; and it would have been necessary to fix for each type of truth and action the degree of probability at which it begins to be reasonable to believe and permissible to act.

M. de Buffon proposait d'assigner une valeur précise à la probabilité très-grande, que l'on peut regarder comme une certitude morale , et de n'avoir au delà de ce terme , aucun égard à la petite possibilité d'un événement contraire. Ce principe est vrai , lorsque l'on veut seulement appliquer à l'usage commun le résultat d'un calcul; et dans ce sens tous les hommes l'ont adopté dans la pratique, tous les philosophes l'ont suivi dans leurs raisonnements : mais il cesse d'être juste , si on l'introduit dans le calcul même , et surtout si on veut l'employer à établir des théories , à expliquer des paradoxes , à prouver ou à combattre des règles générales. D'ailleurs, cette probabilité, qui peut s'appeler certitude morale , doit être plus ou moins grande, suivant la nature des objets que l'on considère , et les principes qui doivent diriger notre conduite ; et il aurait fallu marquer pour chaque genre de vérités et d'actions , le degré de probabilité où il commence à être raisonnable de croire et permis d'agir.

Summarizing . . .

Bernoulli: Very high probability provides practical certainty.

Condorcet: Bernoulli's principle is a bridge from mathematics to practical life.

Cournot: Bernoulli's principle is the only bridge from mathematics to physical reality.

The lottery paradox

According to Cournot's principle,

- In a lottery with many, many tickets, drawing each ticket is physically impossible.
- Given a probability distribution that (1) assigns tiny probability to each of many possibilities or (2) is continuous, each outcome is impossible.

For many moderns, this a decisive argument against Cournot's principle.

Why did Bernoulli, Cournot, and so many others think differently?

Why did Bernoulli, Cournot, and so many others ignore the lottery paradox when equating high probability with moral certainty?

Bayesians sometimes ask the question scornfully.

For example, Diaconis and Skyrms write:

- We cannot help but wonder whether this was to some extent a strategy for brushing off philosophical interpretational problems, rather than a serious attempt to confront them.
- By now, no theorist would be fooled by Bernoulli's swindle.

Ten Great Ideas About Chance, by Persi Diaconis and Brian Skyrms, Princeton University Press, 2018, pp. 66, 67.

For many moderns, the lottery paradox is a decisive argument against Cournot's principle.

Why did Bernoulli, Cournot, and so many others think differently?

Bernoulli and Cournot did not begin with a probability measure *tout fait*.

Bernoulli's theorem gave just one argument. Arguments were combined to make a case. His calculus for the combination was more Dempster-Shafer* than Kolmogorov.

Probability was still constructive for Cournot:

- Rule of compound probability, not definition of conditional probability.
- Probability as becoming, when causal chains are sufficiently entangled.

Lottery paradox was an anomaly.**

* Non-additive probabilities in the work of Bernoulli and Lambert. *Archive for History of Exact Sciences* **19** 309-370.
** Condorcet, Sur la probabilité des faits extraordinaires, Bru and Crépel, pp. 432-448.

The Russian statisticians

Antoine-Augustin Cournot
1801-1877

Exposition de la théorie des chances et des probabilités, 1843

Statistics is usually taken to mean (as the etymology indicates), the collection of facts arising from the clustering of people in civil societies. But for us the word will take on a more extended meaning.

By statistics, we mean the science that collects and systematizes numerous facts of every kind, so as to obtain numerical ratios that are reasonably independent of random anomalies and indicate the existence of regular causes whose influence is combined with that of random causes.

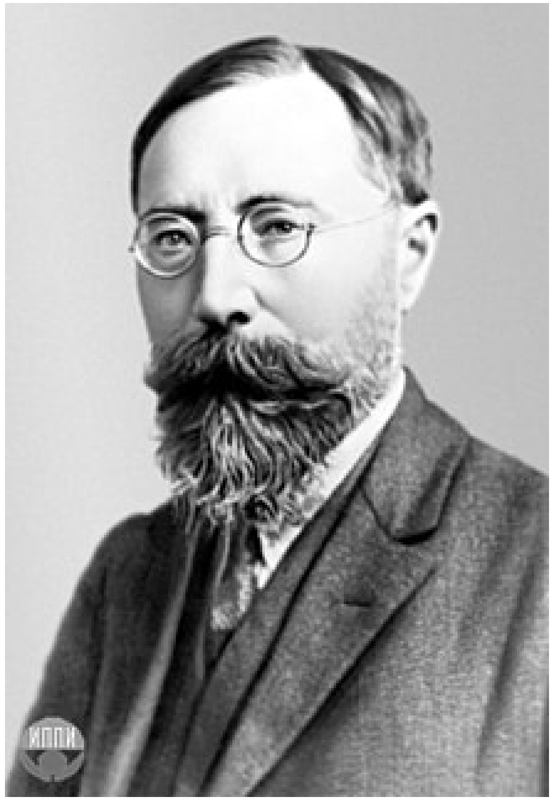
Antoine Augustin Cournot
1801-1877

Cournot’s 1843 book on probability and statistics was a model of clarity and a philosophical gem.

Poisson used “law of large numbers” to name an empirical regularity.

Cournot explained that Poisson’s empirical law can be deduced from

Bernoulli’s theorem
+
Cournot’s principle.



Aleksandr Chuprov
1874-1926

Professor at St. Petersburg.
Left Russia in 1917.

In his 1910 master's thesis in Moscow, Chuprov emphasized Cournot's proof of the law of large numbers.

He called the two ingredients "lemmas":

- Bernoulli's mathematical lemma
- Cournot's logical lemma

Chuprov's version of the logical lemma:

An event of very low probability happens rarely.

Chuprov and his influence

1911	Aleksandr Chuprov	лемма… носит логический характер lemma … has logical character
1925	Evgeny Slutsky	auf einem besonderen Lemma sich begründet based on a particular lemma
1935	Oskar Anderson	Cournotsche Lemma / Cournotsche Brücke Cournot’s lemma / Cournot’s bridge

The French mathematicians

Jacques Hadamard (1865-1963), Emile Borel (1871-1956), and Paul Lévy (1886-1971) had little interest in mathematical statistics.

They wanted Cournot's principle for statistical mechanics, for the second law of the thermodynamics in particular.

When the probability is exceedingly small, the event does not happen.

Statisticians considered larger small probabilities, and Chuprov was content to say that the event happens rarely.

The naming

In 1949, the Swiss journal *Dialectica* published a special issue on Warhscheinlichkeitstheorie und Wirklichkeit. Contributors included Emile Borel, Paul Lévy, and Oskar Anderson.

Padrot Nolfi (1903-1973), co-editor of the special issue, reported on it at the international congress on philosophy of science in Paris later that year.

Fréchet, president of the session on probability theory, included an introduction to the session in the proceedings that appeared in 1951.

Fréchet

Reserve lemme for its use in axiomatic mathematics.

Say principe de Cournot instead of lemme de Cournot.

Fréchet distinguinshed three versions, advocated by different authors in the congress session and *Dialectica* issue.

A. An event with very small probability happens rarely. (Anderson)

B. An event with very small probability is “practically impossible”.

C. When an event has extremely small probability, act as if it should not happen.

Fréchet’s opinions

1. A probability is a physical quantity.

2. Specify the event of small probability in advance.

Fréchet argued that Cournot's principle leads us to regard a probability as a physical quantity, analogous to quantities such as the height of a chair:

probability ~ height

event and a category of trials ~ chair

"… we are led to regard probability as *a physical quantity attached to an event and a category of trials; the probability and the frequency of the event in a large number of trials being approximately equal*."

Dans cette manière de faire, on est conduit à considerer la probabilité comme *une grandeur physique attachée à un événement et à une catégorie d'épreuves et dont les fréquences de cet événement dans un grand nombre d'épreuves sont des mesures approchées*.

Frechet's formulation fails when we consider a stochastic process that can play out only once, as in economics.

Trygve Haavelmo had famously solved this problem in 1944 using Cournot's principle!

The reluctance among economists to accept probability models as a basis for economic research has, it seems, been founded upon a very narrow concept of probability and random variables. Probability schemes, it is held, apply only to such phenomena as lottery drawings, or, at best, to those series of observations where each observation may be considered as an independent drawing from one and the same `population'. From this point of view it has been argued, e.g., that most economic time series do not conform well to any probability model, `because the successive observations are not independent'. But it is *not* necessary that the observations should be independent and that they should all follow the same one-dimensional probability law. It is sufficient to assume that the *whole set* of, say *n*, observations may be considered as *one* observation of *n* variables (or a `sample point') following an *n*-dimensional *joint* probability law, the `existence' of which may be purely hypothetical. Then, one can test hypotheses regarding this joint probability law, and draw inferences as to its possible form, by means of *one* sample point (in *n* dimensions). Modern statistical theory has made progress in solving such problems of statistical inference.

In fact, if we consider actual economic research – even that carried on by people who oppose the use of probability schemes – we find that it rests, ultimately, upon some, perhaps very vague, notion of probability and random variables. For whenever we apply a theory to facts we do not – and we do not expect to – obtain exact agreement. Certain discrepancies are classified as `admissible', others as `practically impossible' under the assumptions of the theory. And the *principle* of such classification is itself a theoretical scheme, namely one in which the vague expressions `practically impossible' or `almost certain' are replaced by `the probability is near to zero', or `the probability is near to one'.

Fréchet on the lottery paradox

…the objection…leads us to spell out what goes without saying…

… When we say an event with extremely small probability is practically impossible, we are making a prediction about an event specified *before* the trial where we see whether or not it happens.

. . . l’objection n’était pas inutile, car elle conduit à préciser ce qui va sans dire, -- mais qui va encore mieux en le disant, -- que, considérant un événement comme pratiquement impossible quand sa probabilité est extrêment petite, nous entendons formuler une prédiction au sujet d’un événement bien défini *avant* qu’ait lieu l’épreuve où l’on constatera si l’événement s’est ou non réalisé.

Specify the event of small probability in advance.

Was Fréchet original in spelling this out?

> In the context of testing, Cournot had already spelled it out, and the issue had been discussed in the English statistics literature (Venn, Edgeworth, Fisher).

Did he convince anyone?

> Probably not. Non-Bayesian statisticians agreed that we need to select a test in advance. But they wanted to preserve the objectivity of probability, and this is threatened if only those high probabilities selected in advance have meaning.

Glenn Shafer's opinion:

> I agree with Fréchet that probabilities are predictions.
>
> But to make proper place for both the subjective and the objective aspects of prediction, we need the adversarial framework of game theory.
>
> > Player I gives probabilities.
> > Player II bets against them.
>
> Both players may be using subjective opinions, but Player I's probabilities gain objective status, in the context of the two players' information, to the extent that they withstand Player II's betting.

Glenn Shafer and Vladimir Vovk, *Game-Theoretic Foundations for Probability and Finance*, Wiley, 2019.

For quotations concerning Cournot’s principle from a hundred scholars over several centuries, see my working paper

"That's what all the old guys said":
The many faces of Cournot's principle

at www.probabililityandfinance.com.